

A NORMAL FORM OF THE NON-LINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this note we discuss normal forms of the completely resonant non-linear Schrödinger equation on a torus, with particular applications to quasi periodic solutions.

1. INTRODUCTION

The aim of this paper is to study an approximate normal form for the completely resonant cubic non-linear Schrödinger equation

$$(1) \quad i v_t - \Delta v = \kappa |v|^2 v,$$

NLS for short, on an n -dimensional torus ($n > 1$) whose coordinates we denote by φ (that is with periodic boundary conditions), with the purpose of applying KAM Theory as for instance in [8], [9],[10]. We shall perform a very detailed study in all dimensions with the most precise results in dimensions 2 and 3.

The NLS in dimension 1 has a long history, it is completely integrable and several explicit solutions are known. Moreover it has a convergent normal form, see [13]. In higher dimension we loose the complete integrability and all the techniques associated to it, but we can start from the well known fact that the NLS (1) is an infinite dimensional Hamiltonian system (see Formula (8)) whose linear part consists of infinitely many independent oscillators with rational frequencies and hence completely resonant (all the bounded solutions are periodic). The presence of the non-linear term couples the oscillators and modulates the frequencies so that one expects small quasi-periodic (and almost-periodic) solutions to exist for appropriate choices of the initial data. To prove the existence of such quasi-periodic solutions for Hamiltonian PDE's there are two main approaches in the literature: one by KAM theory and the other by using Lyapunov-Schmidt decomposition and then Nash-Moser implicit function theorems (the so-called Craig-Wayne-Bourgain method). It is important to notice however that for both approaches it is necessary to start from a suitably non degenerate normal form and the existence of such normal form is not apparent for equation (1).

The object of this paper is to construct an appropriate normal form and show that it satisfies the hypothesis of a KAM algorithm, we will also briefly discuss how our normal form relates to the other main method of producing quasi-periodic solutions.

Before describing our results let us give an extremely sketchy overview of the existing literature on quasi-periodic solutions for the NLS.

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1.1. Background. In the literature on the NLS in higher dimensions, most of the results are restricted to simplified models such as

$$(2) \quad iv_t - \Delta v + M_\sigma v = f(\varphi, |v|^2)v$$

where M_σ is a “Fourier multiplier” i.e. a linear operator, depending on a finite number—say m —of free parameters σ , which commutes with the Laplacian. The role of the “Fourier multiplier” is to ensure that the equation (2), linearized at $v = 0$ has quasi-periodic solutions with m frequencies which excite m Fourier modes (say v_1, \dots, v_m with $v_i \in \mathbb{Z}^n$) and leave the others at rest. The modes v_i on which the linear motion takes place are called the *tangential sites*, and one constructs quasi-periodic solutions of equation 2, which are approximately confined to these Fourier modes.

The first proofs of existence of periodic and quasi-periodic solutions for equation (2) were given by Bourgain, see [6] and [7]. Then Eliasson and Kuksin, in [8], proved both existence and linear stability of quasi-periodic solutions for equations like (2), by using KAM theory. In this setting the main difficulty is to prove measure estimates on the set of initial data for which quasi-periodic solutions occur. More precisely one needs to impose the *second Melnikov condition*, this was done by a subtle analysis of the “Toeplitz Lipschitz” properties of the NLS Hamiltonian.

In the case of dimension $n = 2$ Geng, You and Xu proved the existence of quasi-periodic solutions for equation (1) by a combination of non-integrable normal form, momentum conservation (in the spirit of [9]) and the ideas of Kuksin and Eliasson.

The existence of wave packets of periodic solutions for equation (1) (as well as for the beam equation) in any dimension is proved in [11] and [12]. In those papers the authors were able to deal also with Dirichlet boundary conditions for which the normal form is much more complicated.

1.2. Description of the methods and results. While trying to understand the connections between the result of [10] and [11]–[12], the first author started to understand that one could even attack the much more complicated case of quasi-periodic solutions in case $n > 2$. In particular it became clear that, the challenging problem of completely understanding the NLS Hamiltonian after one step of Birkhoff normal form, had subtle combinatorial and geometric aspects which needed a very careful and non-trivial analysis which is fully performed in this paper.

Resonant Birkhoff normal form is a well-known approach to resonant or degenerate dynamical systems and works very well in the case of the one-dimensional NLS, see [5]. Roughly speaking, consider a Hamiltonian

$$H = H^{(2)}(p, q) + H^{(4)}(p, q), \quad H^{(2)}(p, q) = \sum_k \lambda_k (p_k^2 + q_k^2)$$

where $H^{(4)}$ is a polynomial of degree 4 and the λ_k are all rational.

A step of “resonant Birkhoff normal form” is a symplectic change of variables which reduces the Hamiltonian H to

$$H_N = H^{(2)}(p, q) + H_{res}^{(4)}(p, q) + H^{(6)}$$

where $H^{(6)}$ is an analytic function of degree at least 6 while $H_{res}^{(4)}$ is of degree 4 and Poisson commutes with $H^{(2)}$. Then one wishes to treat $H^{(2)}(p, q) + H_{res}^{(4)}(p, q)$ as the new unperturbed Hamiltonian and $H^{(6)}$ as a small perturbation. This can work provided that

$H^{(2)}(p, q) + H_{res}^{(4)}(p, q)$ is simple enough (possibly completely integrable) and has quasi-periodic motions for large classes of initial data (which now play the role of the parameters σ). An ideal situation is when the λ_k are non-resonant up to order 4 so that the normal form is integrable, for example, in

$$H^{(2)} + H_{res}^{(4)} = \sum_{k=1}^N \lambda_k (p_k^2 + q_k^2) + \sum_{k=1}^m (p_k^2 + q_k^2)^2,$$

the quartic term produces an integrable twist on the first m frequencies. Then one chooses these as tangential sites and passes to action-angle variables $p_k^2 + q_k^2 = \xi_k + y_k$ for $k = 1, \dots, m$. It is easily seen that the anisochronous twist implies that the linear frequencies now depend on the initial datum ξ and hence for almost all ξ the motions are quasi-periodic.

Our setting is quite far from being ideal and one has to start by dividing, in a very careful way, the oscillators into two suitable subsets, the *tangential* and the *normal* sites. This we shall do by imposing the condition that the tangential sites are *generic* (cf. Definition 3.2 and §9.1 for a precise statement), with the strategy of analyzing the equation near the solutions in which the normal sites are at rest and the tangential sites move quasi-periodically on an m -torus.

This is done by *doubling* the action variables in the tangential sites, so each action variable is written as $\xi_i + y_i$ where the ξ_i are treated as parameters (which are used to parametrize the quasi-periodic or almost-periodic solutions obtained by the perturbative method), the y_i instead remain part of the actual symplectic dynamical variables. The introduction of these parameters allows to treat the orbits which produce diffusive phenomena as singularities of the perturbative algorithm (the problem of small denominators). This technique was introduced by Pöschel to prove the existence of lower dimensional tori in finite-dimensional systems and was extended to infinitely-many degrees of freedom in various papers, see [14], [3].

In our case the normal form Hamiltonian $H^{(2)} + H_{res}^{(4)}$ is non-integrable and rather complicated. The structure of this normal form was first discussed by Bourgain in [6] and then revisited in [11] and [12] for the more complicated case of Dirichlet boundary conditions.

The key point in [11] is that, for most of the choices of tangential sites, the leading order of the normal form Hamiltonian is quadratic and block diagonal.

This is not explicit in [11], so let us reformulate the result using, as we shall do in this paper, complex symplectic variables $w_k = (z_k, \bar{z}_k)$ for the normal sites and action angle variables $(\xi + y, x)$ for the tangential sites.

$$(3) \quad H^{(2)} + H_{res}^{(4)} = (\omega(\xi), y) + \frac{1}{2}(y, Ay) - \frac{1}{2}wM(\xi, x)Jw^t + O(w^3)$$

where A is an invertible twist matrix, J is the standard symplectic matrix and $M(\xi, x)$ is a complex matrix such that the quadratic form $Q = -\frac{1}{2}wM(\xi, x)Jw^t$ is real, semi-definite and *block diagonal with blocks of uniformly bounded dimension*. The entries of M depend analytically on the parameters ξ and on the angle variables x and the Hamilton equations for the normal form decouple to (we represent w as a row vector):

$$\dot{y} = 0, \quad \dot{x} = \omega(\xi), \quad \dot{w} = iwM(\xi, x_0 + \omega(\xi)t).$$

In the present paper we largely improve this result by showing that, for generic choices of the tangential sites, not only the quadratic form is block diagonal but that the dimension of the blocks is bounded by $n + 1$; in particular the fact that the dimension of the block

does not depend on the number of tangential sites enables us to consider infinitely many tangential sites, this is crucial if one wants to construct almost-periodic solutions.

1. By relating the normal form to appropriate combinatorial graphs we describe completely and efficiently the structure of the quadratic form; of particular relevance is the fact that the infinite-dimensional quadratic form is described by a finite number of combinatorially defined graphs.

2. The next substantial result, Theorem 1, consists in proving that the normal form Hamiltonian (3) is reducible to constant coefficients:

$$(\omega(\xi), y) - \frac{1}{2} w M'(\xi) J w^t,$$

where the quadratic form is still real but we loose information on the signature of the single blocks. In general whether one may reduce a quadratic Hamiltonian to constant coefficients is a difficult question even for finite dimensional systems. We are moreover in an infinite-dimensional context so that there is also a convergence issue to be treated. Due to our wise choice of the tangential sites however the change of variables which reduces the normal form to constant coefficients is completely explicit and one can check easily all the convergence issues. See [16] for similar results in the context of finite dimensional systems.

3. The final step is to diagonalize the Hamiltonian (by a linear change of variables $U(\xi)$) and obtain:

$$(4) \quad (\omega(\xi), y) + \sum_{|k| > C} \tilde{\Omega}_k(\xi) |z_k|^2 + \tilde{Q}(\xi, w) + \text{Perturbation},$$

where $\omega(\xi), \tilde{\Omega}(\xi)$ are real and $\tilde{Q}(\xi, w)$ is a finite dimensional quadratic form (i.e. it involves only a finite number of normal sites $w_h = u_h, \bar{u}_h$). The matrix $U(\xi)$ and the functions $\omega(\xi), \tilde{\Omega}(\xi)$ are linearly homogeneous analytic functions of ξ for all ξ outside a real algebraic hypersurface. See Theorem 2 for notations and details.

4. We show that in dimension $n = 2$ the non-elliptic terms in (4) vanish for appropriate choices of the parameters ξ_i . As a consequence the non-integrable normal form used in the paper [10] can be reduced to a standard one, Corollary 1. This allows us to identify which of the solutions found in [10] are linearly stable (resp. unstable).

5. In dimension > 2 we then study the question of how to apply an abstract KAM algorithm to our Hamiltonian. We follow [14] and [3]. A main point is to prove a “non-degeneracy condition”, as stated in [14]. We obtain a complete result in dimension 3 and produce a finite (but rather heavy) algorithm to check the property in any dimension. Let us denote by $\tilde{\Omega}_k$ all the eigenvalues of the matrix M' (as we have said it is possible that a finite number is complex valued). We study the set of values ξ , where at least one of Melnikov resonances

$$(5) \quad (\omega(\xi), \nu) = 0, \quad (\omega(\xi), \nu) + \tilde{\Omega}_k(\xi) = 0, \quad (\omega(\xi), \nu) + \tilde{\Omega}_k(\xi) + \sigma \tilde{\Omega}_h(\xi) = 0$$

occur. In the first equality we assume $\nu \neq 0$ while in the third equality we assume that $(\sigma, \nu, h, k) \neq (-, 0, h, h)$ because these give trivial resonances. Since equation (1) has the total momentum as a preserved quantity we only need to consider those choices of $\sigma = +, -, \nu \in \mathbb{Z}^m$ and of normal sites h, k which are compatible with momentum conservation (see Proposition 6.2 item v)).

In Theorem 3 we prove in dimensions 2,3 that the set of ξ which satisfies a non-trivial Melnikov resonance is of measure zero for all choices of (σ, ν, h, k) compatible with momentum conservation.

These Conditions are necessary in order to follow the KAM algorithm as in [14], and sufficient at least at a formal level, since the expression in (5) appear as denominators in the homological equation. Even if at the moment we are unable to prove the second Melnikov condition in dimension > 3 we still show that a slightly weaker statement on separation of eigenvalues, proved in Proposition 6.21 is sufficient to perform the KAM algorithm in our case. Thus although in dimensions ≤ 3 our results are more precise, we still have an answer to the original question in all dimensions. Finally we briefly discuss how our Theorems lead to the convergence of an abstract KAM scheme for ξ in some *Cantor-like set* defined iteratively. To conclude a KAM theorem one should prove that this set is non-empty and of positive measure; we do not discuss here this last question (which can be handled by following [8]). The explicit construction of a KAM iteration scheme and the measure estimates will appear in a forthcoming paper.

If one is willing to give up linear stability results one can probably use the CWB method. Then it is only necessary to check the first two Melnikov conditions and this we do in complete generality (Theorem 3 item 1.).

In our setting, the singularities (i.e. the values for which one of the Melnikov denominators is zero) appear at the loci where the eigenvalues of some matrices depending parametrically (and polynomially) from the ξ_i coalesce or become 0. These loci are algebraic hypersurfaces, and then the full KAM algorithm producing the space of parameters for quasi-periodic solutions converges outside countably many small neighborhoods of these hypersurfaces.

The problem arises in the study of the second Melnikov equation where we have to understand when it is that two eigenvalues become equal or opposite. This is essentially equivalent to using the classical Theory of Sylvester. The condition for a polynomial to have distinct roots is the non-vanishing of the discriminant while the condition for two polynomials to have a root in common is the vanishing of the resultant. In our case these resultants and discriminants are polynomials in the parameters ξ_i so, in order to make sure that the singularities are only in measure 0 sets (in our case even an algebraic hypersurface), it is necessary to show that these polynomials are formally non-zero. This is a purely algebraic problem involving, in each dimension n , only finitely many explicit polynomials (cf. §10) and so it can be checked by a finite algorithm.

The problem is that, even in dimension 3, the total number of these polynomials is quite high (in the order of the hundreds or thousands) so that the algorithm becomes quickly non practical.

In order to avoid this we have experimented with a conjecture which is stronger than the mere non-vanishing of the desired polynomials. We expect our polynomials to be irreducible and separated in a sense explained in §12.

This we prove in dimension 3 (in dimension 2 it is almost immediate), by a mixture of theoretical arguments and a few direct algorithmic verifications. Then this strong result immediately leads to the analyticity of the $\tilde{\Omega}_k(\xi)$ in the parameters ξ and the verification of the second Melnikov condition.

In order to prove irreducibility and separation for all dimensions it would be necessary to eliminate the direct verification of some special cases and thus strengthen the theoretical approach. For the moment this remains conjectural.

Another interesting point is that our reduction algorithm of item **2.** cannot exclude that a finite number of blocks in M may have complex eigenvalues $\tilde{\Omega}_k(\xi)$ for positive ξ . In general, again following Sylvester Theory, one can state the condition that the $\tilde{\Omega}_k \in \mathbb{R}$ as a system of a finite number of explicit polynomial inequalities in the parameters, this

determines an open possibly non-empty region and one has to show that it intersects the positive sector. In dimension $n = 2$ this can be done trivially, but already the case of dimension 3 seems very challenging from a computational point of view.

2. NOTATIONS

2.1. Symplectic formalism. Consider a Nonlinear Schrödinger equation on the torus \mathbb{T}^n (NLS for brevity):

$$(6) \quad iu_t - \Delta u = \kappa |u|^2 u + \partial_{\bar{u}} G(u, \bar{u})$$

where $u := u(t, \varphi)$, $\varphi \in \mathbb{T}^n$ and $G(a, b)$ is a real analytic function whose Taylor series starts from degree 6; notice that we restrict to the case where there is no explicit dependence on the spatial variable φ . It is well known that equation 6, the NLS, is a Hamiltonian equation and has the momentum $\int_{\mathbb{T}^n} \bar{u}(\varphi) \nabla u(\varphi)$ as integral of motion (notice that if $f(u, \bar{u}) = G(|u|^2)u$, then also the L^2 norm $\int_{\mathbb{T}^n} |u(\varphi)|^2$ is preserved).

We shall see that the essential part of the equation is the cubic term. As for the constant κ it can be rescaled to ± 1 . In our paper we set it equal to 1 since the other case is quite similar.

Passing to the Fourier representation

$$(7) \quad u(t, \varphi) := \sum_{k \in \mathbb{Z}^n} u_k(t) e^{i(k, \varphi)}$$

Eq. 6 can be written as an infinite dimensional Hamiltonian dynamical system $\dot{u} = \{H, u\}$ with Hamiltonian

$$(8) \quad H := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{k_1 \in \mathbb{Z}^n : k_1 + k_3 = k_2 + k_4} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} + G(u, \bar{u})$$

on the scale of complex Hilbert spaces

$$(9) \quad \bar{\ell}^{(a,p)} := \{u = \{u_k\}_{k \in \mathbb{Z}^n} \mid \sum_{k \in \mathbb{Z}^n} |u_k|^2 e^{2a|k|} |k|^{2p} := \|u\|_{a,p}^2 \leq \infty\},$$

$$a > 0, \quad p > n/2.$$

with respect to the complex symplectic form $i \sum_k du_k \wedge d\bar{u}_k$.

These choices are rather standard in the literature:

Remark 2.2. The condition imposed on u by (9) means that:

- We restrict our study to functions which extend to analytic functions in the domain of the complex torus $\mathbb{C}^n / 2\pi\mathbb{Z}^n$ where $(z_1, \dots, z_n) \in \mathbb{C}^n$, $\text{Im}(z_i) \leq a$.
- The functions on the boundary are in the Sobolev space H^p .
- The condition $p > n/2$ implies that the function space under consideration embeds in L^∞ . In particular the following uniform bound holds for each $u \in \bar{\ell}^{(\mathbf{a}, \mathbf{p})}$:

$$(10) \quad |u_k| \leq C(s, a) \frac{\|u\|_{a,p} e^{-a|k|}}{\langle k \rangle^{p-n/2}}, \quad \langle k \rangle := \max(1, |k|).$$

In fact this implies that $\bar{\ell}^{(\mathbf{a}, \mathbf{p})}$ has a Hilbert algebra structure.

2.3. Analytic Hamiltonians. We consider real Hamiltonians on the space $\bar{\ell}^{(a,p)}$ which can be formally expanded in Taylor series

$$F = \sum_i F^{(i)}(u, \bar{u}) = \sum_i \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^\infty \\ |\alpha + \beta|_1 = i}} F^{(\alpha, \beta)} \prod_{k \in \mathbb{Z}^n} u_k^{\alpha_k} \bar{u}_k^{\beta_k}$$

where $F^{(i)}$ is a multilinear form of i variables $(u, \bar{u}) \in \bar{\ell}^{(a,p)} \times \bar{\ell}^{(a,p)}$.

We require that the series is totally convergent in some ball of positive radius, so that by definition the function is analytic. Usually this will be verified, using Formula (10), by determining a positive r so that if $\|u\|_{a,p} < r$

$$\sum_i \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^\infty \\ |\alpha + \beta|_1 = i}} |F^{(\alpha, \beta)}| \prod_k \frac{e^{-a(\alpha_k + \beta_k)|k|}}{\langle k \rangle^{(\alpha_k + \beta_k)(p-n/2)}} \|u\|_{a,p}^i < \infty.$$

Clearly the NLS Hamiltonian belongs to this class and is convergent on any ball B_r .

We will also be interested in Hamiltonian vector fields $X_F = \{\partial_{\bar{u}_k} F\}_{k \in \mathbb{Z}^n}$, where F is an analytic Hamiltonian in the ball B_r . With this notation the Hamilton equations are $\dot{u}_k = iX_F$. We will require that the vector field maps $B_r \rightarrow \bar{\ell}_{a,p}$ and we will use the norm

$$|X_F|_r := \sup_{u \in B_r} \|X_F\|_{a,p}/r.$$

notice that if $|X_F|_r < 1/2$ then the Hamiltonian flow is well defined up to $t = 1$ and depends analytically on the initial data so that the symplectic change of variables is well defined and analytic say from $B_{r/2} \rightarrow B_r$. Notice that this condition is NOT verified by $H^{(2)}$ but just by $H^{(4)}$.

2.4. Elliptic-action angle variables. As explained in the introduction we will be interested in non symmetric domains $D \subset B_\epsilon$ where some of the variables u_k (the tangential sites) are bounded away from zero and hence can be passed in action angle variables while all the other variables are much smaller.

We first partition

$$\mathbb{Z}^n = S \cup S^c, \quad S := (v_1, \dots, v_m)$$

where: the set S are called *tangential sites* and S^c the *normal sites*. We then introduce the elliptic action angle coordinates $(\xi + y, x) \times (z, \bar{z}) \in \mathbb{R}^m \times \mathbb{T}^m \times \ell^{(a,p)}$ where $\mathbb{T}^m := \mathbb{R}^m / 2\pi\mathbb{Z}^m$ and we denote by $\ell^{(a,p)}$ the subspace of $\bar{\ell}^{(\mathbf{a}, \mathbf{p})} \times \bar{\ell}^{(\mathbf{a}, \mathbf{p})}$ generated by the indices in S^c and $w = (z, \bar{z})$ (considered as row vectors) are the corresponding coordinates. As explained in the introduction the ξ are positive parameters while the (y, x) are the conjugate dynamical variables. The symplectic form is

$$dy \wedge dx + dz \wedge d\bar{z}.$$

We consider the domain

$$A_\alpha \times D(s, r) :=$$

$$(11) \quad \left\{ \xi : \frac{1}{2}r^\alpha \leq \xi_i \leq r^\alpha \right\} \times \{x, y, w : x \in \mathbb{T}_s^m, |y| \leq r^2, |w|_{a,p} \leq r\}$$

$$\subset \mathbb{R}^m \times \mathbb{T}_s^m \times \mathbb{C}^m \times \ell^{(a,p)}.$$

Here $0 < \alpha < 2, 0 < r < 1, 0 < s$ are parameters. \mathbb{T}_s^m denotes the open subset of the complex torus $\mathbb{T}_\mathbb{C}^m := \mathbb{C}^m / 2\pi\mathbb{Z}^m$ where $x \in \mathbb{C}^m, |\text{Im}(x)| < s$ (cf. Remark 2.2).

Remark 2.5. It is possible and useful to treat also the case $m = \infty$. In this case we use in Formula (11) an exponentially decaying norm $|y|_{a,p}$ and for A_α a condition of the form $\frac{1}{2}r^\alpha \leq \xi_i e^{a|v_i|} \leq r^\alpha$.

Some comments on the choice of this domain are in order. The main point is the inequality in the ξ which is required to determine a domain where the KAM algorithm can be performed outside the singularities to be determined. The inequality on the y and on the w are just auxiliary and only used to keep the computations inside the domains of convergence. The use of the complex domain \mathbb{T}_s^m is motivated by the need to insure that the solutions that we shall find have some analytic behavior.

Given a Banach space E , a function $F(\xi, y, x, w) : A_\alpha \times D(s, r) \rightarrow E$ is said to be analytic in x, y, w if its Taylor-Fourier series in these variables is totally convergent in the domain $A_\alpha \times D(s, r)$. We use as norm the sup-norm

$$\|F\|_{s,r} = \sup_{A_\alpha \times D(s,r)} |F|_E.$$

As for the variables ξ we will require less regularity, namely we use weighted Lipschitz norms

$$\|F\|_{s,r}^\lambda = \|F\|_{s,r} + \lambda \sup_{\xi \neq \eta \in A_\alpha, (y,w) \in D(s,r)} \frac{|F(\eta) - F(\xi)|}{|\eta - \xi|}.$$

In this new set of variables we are also interested in Hamiltonian vector fields

$$X_F = \{\partial_y F, -\partial_x F, Jd_w F\}$$

which map $D(s, r) \rightarrow \mathbb{T}_s^m \times \mathbb{C}^m \times \ell^{(a,p)}$. For such vector fields we will use the norm

$$|X_F|_{s,r} := \sup_{A_\alpha \times D(s,r)} (|\partial_y F| + r^{-2}|\partial_x F| + r^{-1}\|d_w F\|_{a,p})$$

and the corresponding weighted C^1 norm. The Hamilton equations are then

$$\dot{x} = \partial_y F, \quad \dot{y} = -\partial_x F, \quad i\dot{w} = Jd_w F.$$

Of particular interest are the “real quadratic Hamiltonians” in the normal frequencies; we will represent them by matrices with the notation

$$Q(w) := -\frac{1}{2}wMJw^t, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

so that the Hamiltonian vector field is $X_Q = M$. The matrices M which may appear are in the Lie algebra of the group of real symplectic transformations, see Remark 4.12 and Proposition 4.14.

3. MAIN RESULTS

3.1. The Theorems. We are now ready to state our Theorems.

We shall find a finite list (say $P_1(y), \dots, P_N(y)$, where N depends only on n) of non-zero polynomials with integer coefficients depending on $2n$ vector variables $y = (y_1, \dots, y_{2n})$ with $y_i \in \mathbb{C}^n$. We call the P_i *the resonances*, see §9.1.

Definition 3.2. We say that a list of tangential sites $S = \{v_1, \dots, v_m\} \in \mathbb{Z}^{nm}$ is *generic* if for any subset A of S such that $|A| = 2n$, the evaluation of the resonance polynomials at A is non-zero.

If m is finite this condition is equivalent to requiring that S (considered as a point in \mathbb{Z}^{nm}) does not belong to the algebraic hyper-surface where at least one of the resonance polynomials is zero.

We find also a finite list \mathcal{M} of matrices of dimensions $2 \leq k \leq n+1$ with entries polynomials in the elements $\sqrt{\zeta_i}$ for a list ζ_i , $i = 1, \dots, 2n$ of auxiliary variables. We shall denote by $\mathcal{M}(\xi)$ the list of matrices obtained by substituting to the variables ζ_i any $2n$ elements of the list ξ_1, \dots, ξ_m in all possible ways.

Given any $m \in \mathbb{N}$, $0 < \alpha < 2$ and appropriately small s, r , the following holds (cf. 6.2):

Theorem 1. *For all generic choices $S = \{v_1, \dots, v_m\} \in \mathbb{Z}^{nm}$ of the tangential sites, there exists an analytic symplectic change of variables*

$$\Phi : (y, x) \times (z, \bar{z}) \rightarrow (u, \bar{u})$$

from $A_\alpha \times D(s, r) \rightarrow B_{r, \alpha/2}$ with the following properties.

i) the Hamiltonian (8) in the new variables is analytic and has the form

$$H \circ \Phi = (\omega(\xi), y) + \frac{1}{2}(y, A(\xi)y) - \frac{1}{2}wM'(\xi)Jw^t + P(\xi, y, x, w),$$

where

ii) $\omega_i(\xi) = |v_i|^2 - 2\xi_i + 4\sum_j \xi_j$ and the matrix $A(\xi)$ has 1 on the diagonal and 2 off diagonal and hence it is invertible.

iii) The matrix $M'(\xi)$ is a block-diagonal matrix with the following properties: all except a finite number of the blocks are self adjoint; all the blocks are sum of a scalar matrix plus a term chosen from the finite list $\mathcal{M}(\xi)$.

iv) The perturbation P is of the form $P(\xi, y, x, w) = P^{(3)}(\xi, x, y, w) + P^{(6)}(\xi, x, y, w)$, where— setting $\lambda = r^\alpha / \max_i(|v_i|^2)$ — we have the bounds:

$$\|P^{(3)}(\xi, x, y, w)\|_{s,r}^\lambda \leq Cr^{3+\alpha/2} \quad \|P^{(6)}(\xi, x, y, w)\|_{s,r}^\lambda \leq Cr^{\max(5\alpha, 1+\frac{5}{2}\alpha)},$$

moreover $P^{(3)}(\xi, x, y, w)$ is at least cubic in w .

Completely explicit statements will appear in §6.

Outline of the proof. The symplectic change of variables Φ is the composition of three steps: first we perform one step of Birkhoff normal form passing to the Hamiltonian (21), then we pass to the elliptic action angle variables and obtain the Hamiltonian (25). We study the block form of the quadratic part of this Hamiltonian, see (34), and produce a list of constraints on the tangential sites so that (34) is as simple as possible, this is the core of the proof. Then we perform a simple but non-perturbative symplectic change of variables, see (44), so that the quadratic Hamiltonian (34) is reduced to constant coefficients. \square

When we apply the theory of normal forms for quadratic Hamiltonians to M' we obtain.

Theorem 2. *There exists a real algebraic hypersurface \mathfrak{A} such that the following holds. There exists a linear change of variables in the (z, \bar{z}) depending analytically on ξ for all $\xi \in A_\alpha \setminus \mathfrak{A}$, such that in the new variables the Hamiltonian is*

$$(12) \quad H_{\text{fin}} = (\omega(\xi), y) + \frac{1}{2}(y, Ay) + \sum_{k \in S^c \setminus \mathcal{H}} \tilde{\Omega}_k |z_k|^2 - \frac{1}{2}wRJw^t + P^{(3)}(\xi, x, y, w) + P^{(6)}(\xi, x, y, w),$$

where:

i) \mathcal{H} is a finite subset of S^c and R is a matrix with only a finite number of non zero entries.

ii) Let us call $\tilde{\Omega}_k$ with $k \in \mathcal{H}$ the eigenvalues¹ of R , we have

$$\tilde{\Omega}_k = |k|^2 - \sum_i A_k^{(i)} |v_i|^2 + \lambda_k(\xi), \quad \forall k \in S^c$$

The $A_k^{(i)}$ are integers with $\sum_i |A_k^{(i)}| < 2n$ and the correction $\lambda_k(\xi)$ is chosen in a finite list, say

$$(13) \quad \lambda_k(\xi) \in \{\lambda^{(1)}(\xi), \dots, \lambda^{(K)}(\xi)\}, \quad K := K(n, m),$$

of different analytic functions of ξ .

ii) The functions $\lambda^{(i)}(\xi)$ are homogeneous of degree one in ξ . This implies that for $\xi \in A_\alpha \setminus \mathfrak{A}_\varepsilon$ where \mathfrak{A}_ε is the spherical neighborhood of \mathfrak{A} of radius $\varepsilon = r^\alpha/4$ the $\lambda^{(i)}(\xi)$ satisfy the bounds

$$(14) \quad |\lambda^{(i)}(\xi)| \leq Cr^\alpha, \quad cr^\alpha \leq |\lambda^{(i)}(\xi) \pm \lambda^{(j)}(\xi)| \leq Cr^\alpha, \quad |\nabla_\xi \lambda^{(i)}(\xi)| \leq C.$$

iii) For $\xi \in A_\alpha \setminus \mathfrak{A}_\varepsilon$ item v) of Theorem 1 holds.

Remark 3.3. Notice that we may further restrict the set \mathcal{H} by requiring that all the block matrices in R are either non-diagonalizable or have all complex eigenvalues. Indeed all the other blocks may be put in the elliptic normal form by a symplectic change of variables. In dimensions 2, 3 we show that also R is diagonalizable for generic values of ξ over the complex numbers. This gives an hyperbolic normal form where for each complex eigenvalue $\tilde{\Omega}_k$ one has a two-by-two diagonal block with $\text{Re}(\tilde{\Omega}_k)I + \text{Im}(\tilde{\Omega}_k)J$, where I is the identity and J the symplectic matrix.

Corollary 1. For $n = 2$ and for appropriate choices of the tangential sites S there exists a cone-like domain D , with $\text{meas}(D \cap A_\alpha) \sim r^\alpha$, such that for all $\xi \in D$ the set \mathcal{H} in Theorem 2 is empty. On $A_\alpha \setminus D$ the set \mathcal{H} is non-empty and such that R is in hyperbolic normal form (see Remark 3.3).

Theorem 3. 1. The set of parameter values ξ for which the first two Melnikov resonances in (5) occur has zero measure.
2. For $n = 2, 3$ the set of parameter values ξ such that the three Melnikov resonances (5) occur is a zero measure set (and for each condition it is algebraic).

Remark 3.4. To prove this theorem it will be essential that the NLS equation (1) preserves the total momentum, this implies that the (σ, ν, h, k) in (5) must satisfy a momentum conservation relation, see Proposition 6.2.

The proof is based on a careful analysis of the characteristic polynomials of the matrices in \mathcal{M} (cf. theorem 1, iv)) and is carried out in sections 11 and 12.

Remark 3.5. In dimension > 3 we shall prove a weak form of the second Melnikov resonance condition (cf. Proposition 6.21). In this form the eigenvalues may have multiplicities but outside of a set of measure zero these multiplicities are finite and uniformly bounded. Moreover the eigenspace for any given eigenvalue is isotropic, it pairs under Poisson bracket only with the eigenspace for the opposite eigenvalue. This is enough to perform a KAM algorithm by solving an appropriate homological equation, see §6.13

¹naturally the matrix may not be diagonalizable

The case $m = \infty$. There are infinitely many infinite sets S which satisfy the non-resonance conditions (§9.1). For these choices of tangential sites most of the previous statements hold verbatim. Some of the quantitative estimates need a more careful analysis.

3.6. Quasi-periodic solutions for the NLS equation (1). As we have stated in the introduction there are two main methods for finding quasi-periodic solutions for non-linear PDE's near to an elliptic fixed point. Let us briefly review how our results on normal form relate to these methods. The technical issues involved in applying these methods for the completely resonant NLS will be analyzed in a separate paper.

1. The Craig–Wayne–Bourgain method

The CWB method is based on a Liapunov–Schmidt decomposition combined with a generalized Newton method to overcome the small divisor problems. We follow the approach in [4] which provides both an extension of these results and a clear exposition using the Nash–Moser approach.

We look for a solution of (1) of the form $v(t, \varphi) = u(\omega t, \varphi)$ where $u : \mathbb{T}^m \times \mathbb{T}^n \rightarrow \mathbb{C}$ lives in a Sobolev space of functions with a Hilbert algebra structure.

Our theorems 1, 2 imply that the bifurcation equation for (1) admits a solution of the form $u_0(x, \varphi) + O(\xi^{3/2})$ where

$$u_0 = \sum_{i=1}^m \sqrt{\xi_i} e^{i(x_i + v_i \cdot \varphi)}.$$

Consider the bifurcation equation linearized at u_0 and written in the Fourier basis both in the space variables φ and in the angles x , this equation is represented by an infinite matrix say L . From theorems 1, 2 L is block diagonal and the blocks are very simply associated to the blocks of M' . Moreover the eigenvalues of L are $\omega(\xi) \cdot \nu \pm \tilde{\Omega}_k(\xi)$ with $\nu \in \mathbb{Z}^m, k \in S^c$. By Theorem 3 item 1. the eigenvalues are not identically zero.

Notice finally that since the matrix is block diagonal (with blocks of dimension $\leq n+1$) its invertibility implies invertibility in any norm with the same bounds, and one may impose that

$$\|L^{-1}u\|_s \leq C\|u\|_{s+\tau}$$

where $\|v\|_s$ is a Sobolev norm in the space and angle variables. From this point we expect to be able to follow the same scheme as in [4], with only small technical variations.

2. KAM theory: let us first restrict to the case $n = 2, 3$ where, by Theorem 3 item 2, the second Melnikov conditions hold. The case $n = 2$ is essentially already covered by [10]. Our analysis produces several improvements in their results, in particular for the parameter values $\xi \in D$ we show that the quasi-periodic solutions obtained in [10] are linearly stable while for $\xi \in A_\alpha \setminus D$ the solutions are unstable.

In the literature (see for instance [3]) *abstract KAM schemes* are based on three main assumptions:

1. A *smallness condition* on the perturbation P , in our case this is Theorem 2 item iii).
2. A *regularity condition*, namely $\omega(\xi)$ must be a diffeomorphism and $\tilde{\Omega}_k(\xi) - |k|^2$ must be a bounded Lipschitz function (we have analyticity and the bounds (14), by Theorem 2).
3. A *non-degeneracy condition*, that is the three Melnikov conditions 5 which in our case are proved in Theorem 3, see also the discussion in §6.13.

As we already mentioned a full discussion of the KAM algorithm will appear elsewhere.

The abstract KAM schemes produce an analytic change of variables, depending on ξ , $\Phi = e^{ad(F)} : D(s/4, r/4) \rightarrow D(s, r)$ such that

$$e^{ad(F)} \circ H = (\omega^\infty(\xi), y^\infty) + \sum_{k \in S^c \setminus \mathcal{H}} \Omega_k^\infty(\xi) |z_k^\infty|^2 - \frac{1}{2} (w^\infty)^t R_\infty w^\infty + P^\infty,$$

where ω^∞ and Ω^∞ are parameters to be determined and R_∞ has the same structure and is simultaneously diagonalizable with R' . Finally

$$P^\infty = \sum_{i,j: 2i+j>2} P_{ij}^\infty(x^\infty)(y^\infty)^i(w^\infty)^j.$$

In the new variables one immediately shows the existence of the quasi-periodic solution

$$y^\infty = 0, \quad w^\infty = 0, \quad x^\infty = x_0^\infty + \omega^\infty t.$$

Φ will be defined for ξ in some complicated Cantor-like set. This part of the algorithms can be performed in our case by following almost verbatim the KAM Theorem 5.1 of [3].

The final issue is to analyze this set, show that it is non-empty and give lower bounds on its measure. To give a flavor of the type of computations required consider the Cantor set \mathcal{C} , which appears at the first step of the algorithm, defined by

$$|(\omega(\xi), \nu)| > \frac{\gamma(A + r^\alpha)}{|\nu|^{\tau_0}}, \quad |(\omega(\xi), \nu) + \tilde{\Omega}_k(\xi)| > \frac{\gamma(A + r^\alpha)}{|\nu|^{\tau_0}},$$

$$(15) \quad |(\omega(\xi), \nu) + \tilde{\Omega}_k(\xi) + \sigma \tilde{\Omega}_h(\xi)| > \frac{\gamma(A + r^\alpha)}{|\nu|^{\tau_0}}$$

for all non-trivial resonances which are compatible with momentum conservation, moreover $A = 1$ if the corresponding function on the left hand side is zero at $\xi = 0$ and $A = 0$ otherwise.

Corollary 2 (Of Theorems 2, 3). *For $n = 2, 3$ and for appropriate choices of τ_0 and γ , the set \mathcal{C} has positive measure in $A_\alpha \setminus \mathfrak{A}_\varepsilon$.*

For a sketch of the proof see §6.15.

This shows that the first step of the KAM algorithm produces a set of positive measure where the desired symplectic change of variables is well defined. As shown by [8], in order to give similar estimates at all steps of the KAM iteration one needs to use the *Töplitz-Lipschitz property* of the NLS Hamiltonian. We do not discuss this last property in the present paper. Notice however that in Theorem 1 and 2 even though the Hessian matrix $\partial_{z_k} \partial_{\bar{z}_h} (P^{(3)} + P^{(6)})$, is not a Töplitz matrix it still satisfies *Töplitz-Lipschitz properties* (as is discussed in detail in [10] for the case $n = 2$). We expect to obtain the measure estimates by following [8] and [10].

The case $n > 3$ is discussed more in detail in subsection 6.18.

Part 1. The study of the dynamics

4. PREPARATION

4.1. **Conservation laws.** Recall the laws of Poisson brackets:

$$(16) \quad \{iu_k, \bar{u}_h\} = \delta_{h,k}, \quad \{u_k, u_h\} = \{\bar{u}_k, \bar{u}_h\} = 0;$$

hence

$$(17) \quad \{u_h \bar{u}_k, u_j\} = i\delta_{j,k} u_h, \quad \{u_h \bar{u}_k, \bar{u}_j\} = -i\delta_{h,j} \bar{u}_k$$

in particular

$$(18) \quad \{|u_k|^2, u_h\} = i\delta_{h,k}u_h, \quad \{|u_k|^2, \bar{u}_h\} = -i\delta_{h,k}\bar{u}_h$$

Definition 4.2. We set

$$(19) \quad M := \sum_{k \in \mathbb{Z}^n} k|u_k|^2, \quad \text{momentum.}$$

A Hamiltonian defined in $\bar{\ell}^{(a,p)}$ which Poisson-commutes with $M = \sum_{k \in \mathbb{Z}^n} k|u_k|^2$ satisfies the constraint of *conservation of momentum*:

$$\text{If } F^{(\alpha,\beta)} \neq 0, \quad \text{one has } \sum (\alpha_k - \beta_k)k = 0;$$

A Hamiltonian defined in $\bar{\ell}^{(a,p)}$ is *even* if it is sum of monomials of even degree: Namely if $F^{\alpha,\beta} \neq 0$ then $\alpha + \beta$ is even.

4.3. Normal form. The quadratic part $H^{(2)} := \sum_k |k|^2 |u_k|^2$ of 8 is an infinite string of harmonic oscillators with all rational frequencies so that the system is *completely resonant* (all the bounded solutions are periodic).

Using the conventions of Lie Theory we shall always denote by $ad(F)$ the operator of Poisson bracket $X \mapsto \{F, X\}$.

For small u (i.e. $\|u\|_{a,p} < \epsilon \ll 1$) we perform a standard step of “resonant” Birkhoff normal form removing all the terms of order four of H which do not Poisson-commute with the quadratic part, see also Remark 4.7.

In fact, by (17), $u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}$ is an eigenvector with respect to $\{H^{(2)}, -\}$ with eigenvalue $i(|k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2)$. Thus we perform the symplectic change of variables $H \mapsto e^{ad(F)}(H)$, generated by the flow of

$$(20) \quad F := -i \sum_{\substack{k_i: k_1 - k_2 + k_3 - k_4 = 0 \\ |k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2 \neq 0}} \frac{u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}}{|k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2}.$$

For ϵ sufficiently small, this is a well known analytic change of variables (cf. [7],[6],[2]) $\bar{\ell}^{(a,p)} \supset B_\epsilon \rightarrow B_{2\epsilon} \subset \bar{\ell}^{(a,p)}$ (where B_ϵ denotes as usual the open ball of radius ϵ) which brings (8) to the form:

$$(21) \quad H_N := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{k_1 + k_3 = k_2 + k_4 \\ |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} + P^{(6)}(u)$$

where $P^{(6)}(u)$ is analytic of degree at least 6 in u and on the ball B_ϵ it is bounded by $C\epsilon^6$ with C a suitable constant. Since we will take ϵ small, $P^{(6)}(u)$ is small with respect to the terms of degree 2, 4 which are bounded by $C_1\epsilon^2, C_2\epsilon^4$ respectively (cf. §4.8). Notice that F commutes with M so that H_N still satisfies momentum conservation, moreover F is even and hence H_N is still even.

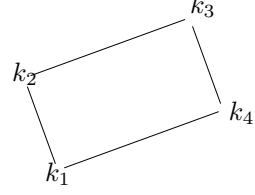
Denote by $\mathcal{P}' := \{(k_1, k_2, k_3, k_4) \mid k_1 + k_3 = k_2 + k_4, |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2\}$.

Trivial computations show that the condition

$$k_1 + k_3 = k_2 + k_4, \quad |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2$$

is equivalent to

$$(22) \quad k_1 + k_3 = k_2 + k_4, \quad (k_1 - k_2, k_3 - k_2) = 0$$



In this set the integer vectors k_1, k_2, k_3, k_4 form the vertices of a *rectangle*. We want to put in evidence the terms where the rectangle degenerates to a segment. Thus define \mathcal{P} to be the subset of \mathcal{P}' where the rectangles are non degenerate. We obtain:

$$(23) \quad H_N = \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_k |u_k|^4 + 2 \sum_{k_1 \neq k_2} |u_{k_1}|^2 |u_{k_2}|^2 + \sum_{(k_1, k_2, k_3, k_4) \in \mathcal{P}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} + P^{(6)}(u)$$

4.4. Choice of the tangential sites. Let us now partition

$$\mathbb{Z}^n = S \cup S^c, \quad S := (v_1, \dots, v_m),$$

where: The set S are called *tangential sites* and S^c the *normal sites*.

Constraint 1. S is a finite set (say $|S| = m$) such that, given any three distinct elements $v_1, v_2, v_3 \in S$, one has that $(v_1 - v_2, v_3 - v_2) \neq 0$.

Remark 4.5. We shall discuss later the extension to S infinite.

In the following we will consider various other constraints on S in order to obtain the simplest possible expression for the Hamiltonian H , with respect to this choice.

Our aim is to study the NLS near the tori associated to the oscillators v_i keeping the other oscillators constant in time at $u_k = 0$. The terms of order four in the normal form introduce what is called a *twist*. That is an anisochronous term such that the frequency depends on the initial datum $|u_{v_i}|^2 = \xi_i$ and $u_k = 0$ for $k \in S^c$.

Let us now set

$$u_k := z_k \text{ for } k \in S^c, \quad u_{v_i} := \sqrt{\xi_i + y_i} e^{ix_i} \text{ for } i = 1, \dots, m;$$

this is a well known symplectic change of variables which puts the tangential sites in action angle variables $(y; x) = (y_1, \dots, y_m; x_1, \dots, x_m)$ close to the action $\xi = \xi_1, \dots, \xi_m$, which we now consider as parameters for the system. The symplectic form is now $dy \wedge dx + i \sum_{k \in S^c} dz_k \wedge d\bar{z}_k$.

It is convenient to think of the z_k, \bar{z}_k as a vector w and denote, for a function of w by $\frac{\partial F}{\partial w}$ the gradient vector (which we think of as a *column*). Further denote by J the infinite

skew symmetric matrix $\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$ where the first block is over the basis z_k and the second over \bar{z}_k . Poisson bracket with respect to this form is

$$(24) \quad \{F, G\} = \frac{\partial F}{\partial y} \cdot \frac{\partial G}{\partial x} - \frac{\partial G}{\partial y} \cdot \frac{\partial F}{\partial x} + i \left(\frac{\partial F}{\partial w} \cdot J \frac{\partial G}{\partial w} \right)$$

By constraint 1, the Hamiltonian 21 can be written as

$$(25) \quad H = H_0 + P^{(3)}(z, y; \xi, x) + P^{(6)}(z, y; \xi, x), \quad \text{with}$$

$$\begin{aligned}
H_0 := & \sum_{i=1}^m (|v_i|^2(\xi_i + y_i) + (\xi_i + y_i)^2) + 4 \sum_{i < j} (\xi_i + y_i)(\xi_j + y_j) \\
& + 4 \sum_{i; k \in S^c} (\xi_i + y_i)|z_k|^2 + \sum_{k \in S^c} |k|^2|z_k|^2 + 4 \sum_{i \neq j; h, k \in S^c}^* \sqrt{(\xi_i + y_i)(\xi_j + y_j)} e^{i(x_i - x_j)} z_h \bar{z}_k \\
& + 2 \sum_{i < j; h, k \in S^c}^{**} \sqrt{(\xi_i + y_i)(\xi_j + y_j)} e^{-i(x_i + x_j)} z_h z_k + 2 \sum_{i < j; h, k \in S^c}^{**} \sqrt{(\xi_i + y_i)(\xi_j + y_j)} e^{i(x_i + x_j)} \bar{z}_h \bar{z}_k
\end{aligned}$$

Definition 4.6. Here \sum^* denotes that $(h, k, v_i, v_j) \in \mathcal{P}$:

$$\{(h, k, v_i, v_j) \mid h + v_i = k + v_j, |h|^2 + |v_i|^2 = |k|^2 + |v_j|^2\}.$$

and \sum^{**} , that $(h, v_i, k, v_j) \in \mathcal{P}$:

$$\{(h, v_i, k, v_j) \mid h + k = v_i + v_j, |h|^2 + |k|^2 = |v_i|^2 + |v_j|^2\}.$$

The term $P^{(3)}$ collects all terms in which at least 3 indices k are in S^c and it is of degree at least three (and at most four) in z, \bar{z} . Recall that we have assumed (constraint 1) that no non degenerate rectangles contain 3 elements of S .

Remark 4.7. Notice that, once we have fixed the tangential sites, we have some freedom in the choice of the Birkhoff normal form transformation F in (20). Indeed one may choose first the tangential sites and then choose F as follows:

$$F := -i \sum_{\substack{k_i: |\{k_1, k_2, k_3, k_4\} \cap S^c| \leq 2 \\ k_1 - k_2 + k_3 - k_4 = 0 \quad |k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2 \neq 0}} \frac{u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4}}{|k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2}.$$

This normal form transformation (taken from [10]) does not change the results in any way (it only changes in a trivial manner the definitions of $P^{(3)}$ and $P^{(6)}$) and may simplify the study of the *Töplitz-Lipschitz* properties.

Conservation laws. Momentum. The conservation of M in the new variables implies that the monomials appearing in H are of the form

$$(26) \quad z^\alpha \bar{z}^\beta y^c e^{i(x, \nu)}, \quad \sum_i v_i \nu_i + \sum_{k \in S^c} (\alpha_k - \beta_k) k = 0$$

where $\nu = (\nu_1, \dots, \nu_m)$, $\nu_i \in \mathbb{Z}$ and α, β are multi-indices in \mathbb{N} .

Parity. In the new variables a Hamiltonian is *even* if $\sum_i \nu_i$ is even when the total degree in w is even and odd otherwise.

Constraint 2. We may further choose the v_i so that $\sum_i \nu_i v_i \neq 0$ when $\sum_i |\nu_i| < 10$, $\nu \neq 0$.

This will imply that $P^{(6)}$ at $z = 0$ does not contain any term of degree 6 or 8 in the elements u_{v_i} and non constant in x .

4.8. Analytic functions and weight decomposition. We want to assign *weights* to all the variables, which keep track of their original definition.

Thus we give weight 2 to the variables y , weight $2 > \alpha > 0$ to ξ and weight 1 to the w .

We work in the domain

$$\begin{aligned}
& A_\alpha \times D(s, r) := \\
& \{\xi : \frac{1}{2} r^\alpha \leq |\xi| \leq r^\alpha\} \times \{x, y, w : x \in \mathbb{T}_s^m, |y| \leq r^2, |w|_{a,p} \leq r\}
\end{aligned}$$

$$\subset \mathbb{R}^m \times \mathbb{T}_s^m \times \mathbb{C}^m \times \ell^{(\mathbf{a}, \mathbf{p})}.$$

Here $0 < \alpha < 2, 0 < r < 1, 0 < s$ are parameters. \mathbb{T}_s^m has been defined in §2.4.

We denote by $\ell^{(\mathbf{a}, \mathbf{p})}$ the subspace of $\bar{\ell}^{(\mathbf{a}, \mathbf{p})} \times \bar{\ell}^{(\mathbf{a}, \mathbf{p})}$ generated by the indices in S^c and $w = (z, \bar{z}) \equiv (z^+, z^-)$ are the corresponding coordinates.

Remark 4.9. We shall write $z_k^\sigma, \sigma = \pm$ when we do not want to specify if we are using z_k or \bar{z}_k .

Remark 4.10. If $r^\alpha < \epsilon$ the domain $A_\alpha \times D(s, r)$ is contained in B_ϵ so that the Hamiltonian is well defined and analytic.

Under this change of variables the total degree of a monomial is preserved, provided we give to y degree (or weight) 2. In the estimates we also give to ξ weight α . This implies (by Formula (11)) that $|\xi^a y^i w^j|_{s,r,\mathbb{C}} \leq r^d$ where the degree d equals the weight $a\alpha + 2i + j$ of the monomial (notice that a can be a half-integer).

Given a Banach space E we consider analytic functions $F : D(s, r) \rightarrow E$. By definition F is analytic if its Taylor-Fourier series in $x; y, w$ is totally convergent. We choose by definition, as norm $|F|_{s,r}$, of an analytic function on $D(s, r)$:

$$(27) \quad |F|_{s,r,E} := \sup_{A_\alpha \times D(s,r)} \|F(\xi, x; y, w)\|_E.$$

4.11. Weight decomposition of formal polynomials. We call V^∞ the space of formal infinite polynomials in y, w with coefficients in $L^2(\mathbb{T}^m, \mathbb{C})$ and F^∞ the subspace formed by the functions which satisfy 26. Here we use the weight only in these variables and keep the ξ as parameters.

We call $V^{(i,j)}$ the subspace of functions of degree i in y and j in w (hence of weight $2i + j$). We call V^a the subspace of functions of weight a and $V^{>a}$ (resp. $V^{<a}$) the subspace of functions of weight $> a$ (resp. $< a$):

$$V^\infty = \oplus_a V^a, \quad V^a = \oplus_{i,j: 2i+j=a} V^{i,j}, \quad V^{>a} = \oplus_{b>a} V^b.$$

We use the spaces of weight ≤ 2 :

- $V^0(x) := L^2(\mathbb{T}^m, \mathbb{C})$,
- $V^{1,0}(x)$ the space of elements $\sum_{i=1}^m f_i(x, \xi) y_i$, thought of as m dimensional column vectors with entries in $L^2(\mathbb{T}^m, \mathbb{C})$ (and depending on parameters ξ),
- $V^{0,1}(x)$ the space of linear forms $\sum_{i \in \mathbb{Z}^n} (f_i(x, \xi) z_i + e_i(x, \xi) \bar{z}_i)$, thought of as ∞ -dimensional column vectors with entries in $L^2(\mathbb{T}^m, \mathbb{C})$ (and parameters ξ) and finally
- $V^{0,2}$ the space of quadratic forms

$$\frac{1}{2} \left(\sum_{i,j \in S^c} q_{(i,+),(j,+)} z_i z_j + q_{(i,+),(j,-)} z_i \bar{z}_j + q_{(i,-),(j,+)} \bar{z}_i z_j + q_{(i,-),(j,-)} \bar{z}_i \bar{z}_j \right).$$

It will be convenient to represent the elements $Q(w) \in V^{0,2}$ as associated to (symplectic) matrices, where

$$(28) \quad Q_M(w) := -\frac{1}{2}(w, M J w)$$

and M has entries $M_{(k,\sigma),(h,\tau)} = \tau q_{(k,\sigma),(h,-\tau)}$.

Remark 4.12. By definition MJ is symmetric, if moreover $Q_M(w)$ is real we have that $M\Sigma$ is self-adjoint, where

$$\Sigma = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}.$$

Remark 4.13. Notice that we have defined V^∞ as complex functions. However our Hamiltonian is real, and clearly V^∞ contains a subspace of real functions. Indeed all our arguments and symplectic changes of variables will be real and hence preserve the real subspace.

The dense subspace $V_{an}^{0,2}$ of $V^{0,2}$ formed by analytic functions is closed under Poisson bracket and we think of it as a smooth form of the Lie algebra of the infinite symplectic group. The space $V_{an}^{0,1}$ is then the standard symplectic space, under Poisson bracket, over which $V_{an}^{0,2}$ acts. In particular we have

Proposition 4.14. *If $f, g \in V_{an}^{0,1}$ and A, B are two elements of the Lie algebra of the symplectic group of $V_{an}^{0,1}$*

$$(29) \quad \{f, g\} = i f^t J g, \quad \{Q_A(w), f\} = i A f, \quad \{Q_A(w), Q_B(w)\} = i Q_{[A, B]}.$$

In particular the evolution of w , defined by Q_A is

$$(30) \quad \dot{w} = i w A$$

We now require conservation of momentum and parity.

In order to stress this

Definition 4.15. We denote the subspaces of $V^0, V^{(1,0)}, V^{(0,1)}, V^{(0,2)}$ which satisfy conservation of momentum and parity by $F^0, F^{(1,0)}, F^{(0,1)}, F^{(0,2)}$. The direct sum of these spaces we denote by $F^{\leq 2}$. In general we denote by $F^{(i,j)}$ the subspace of $V^{(i,j)}$ which satisfies (26) and parity.

Remark 4.16. i) $F^{(0,1)}$ has as basis the elements

$$(31) \quad (\nu, +) \rightarrow e^{i \sum_j \nu_j x_j} z_k, \quad (\nu, -) \rightarrow e^{-i \sum_j \nu_j x_j} \bar{z}_k;$$

$$\sum_j \nu_j v_j + k = 0, \quad \sum_i \nu_i = \text{odd}.$$

ii) In the same way $F^{(0,2)}$ has as basis the products of elements of $F^{0,1}$. That is we have a surjective linear map $b : F^{0,1} \otimes F^{0,1} \rightarrow F^{0,2}$.

Under this map $b[(\nu, \sigma) \times (\mu, \tau)] = b[(\nu', \sigma') \times (\mu', \tau')]$ if and only if $\sigma\pi(\nu) = \sigma'\pi(\nu')$, $\tau\pi(\mu) = \tau'\pi(\mu')$ and $\sigma\nu + \tau\mu = \sigma'\nu' + \tau'\mu'$. The image of these product form a basis of $F^{0,2}$. Explicitly (cf. 4.9) the elements

$$e^{i \sum_j \nu_j x_j} z_k^\sigma z_h^\tau, \quad \sum_j \nu_j v_j + \sigma k + \tau h = 0.$$

4.17. Quadratic normal forms. In the space $F^{\leq 2}$ we will be particularly interested in the subspace of “normal forms” i.e. the Hamiltonians of the form

$$N := (\omega, y) - \frac{1}{2} w Q(x) J w^t,$$

for some choice of the frequency $\omega \in \mathbb{R}^m$ and of the matrix $Q(x)$, both possibly depending on the parameters ξ .

The core of the KAM algorithm is to study the action of the operator $ad(N)$ on $F^{\leq 2}$.

Definition 4.18. Given $R \in F^{\leq 2}$, the equation $ad(N)F = R$ for $F \in F^{\leq 2}$ is called *the homological equation*.

Lemma 4.19. *The operator $ad(N) := x \mapsto \{N, x\}$ acting on $F^{\leq 2}$ can be represented as the block matrix:*

$$(32) \quad \begin{pmatrix} \omega \cdot \partial_x & 0 & 0 & 0 \\ 0 & \omega \cdot \partial_x & 0 & 0 \\ 0 & 0 & \omega \cdot \partial_x + iQ & 0 \\ 0 & i\nabla_x Q & 0 & \omega \cdot \partial_x + i[Q, \cdot] \end{pmatrix},$$

Here, by abuse of notations, $\omega \cdot \partial_x = \sum_{i=1}^m \omega_i \frac{\partial}{\partial x_i}$, is the operator on the entries of the vector or matrix, i.e. the *scalar* operator times the corresponding identity matrix. We have used the basis 4.16 for $F^{0,1}$ and the matrix representation for $F^{0,2}$.

Lemma 4.20. *If for $F^{0,2}$ we use as basis the products of the elements of the basis in $F^{0,1}$, then by the Leibniz rule then $ad(N)$ on $F^{0,2}$ is induced, under the map $b : F^{0,1} \otimes F^{0,1} \rightarrow F^{0,2}$ (cf. Remark 4.16) by the matrix*

$$(33) \quad ad(N)|_{F^{0,1}} \otimes I + I \otimes ad(N)|_{F^{0,1}}.$$

Proof of 4.19, 4.20. We just apply the rules of Poisson brackets discussed in Proposition 4.14. \square

4.21. Final form for the Hamiltonian. By definition an analytic function on $D(s, r)$ can be Taylor expanded in y, w to obtain an element of V^∞ ; given $F \in V^\infty$ we will denote by $F^{(i,j)}$ the projection of F on $V^{(i,j)}$, same for all the other subspaces.

For r small enough, $H : D(s, r) \rightarrow \mathbb{R}$ is analytic, so it is an element of V^∞ . We obtain a formal polynomial whose monomials are of the form $m(x)\xi^a y^i w^j$ where a can have half-integer coordinates (this corresponds to a term in u of degree $2(i+a)+j$).

We drop in formula (25) the constant part (depending only on the parameters ξ) and separate $H = N + P$ where $N = H_0^{\leq 2} \in V^2$ is a “normal form” and $P = H_0^{>2} + P^{(3)} + P^{(6)}$ is *small* with respect to N .

We obtain, with the notation of Formula (28)

$$(34) \quad N := (\omega(\xi), y) + \sum_k \Omega_k |z_k|^2 + Q_M(w) := D + Q_M(w),$$

where $D := (\omega(\xi), y) + \sum_k \Omega_k(\xi) |z_k|^2$ and

$$(35) \quad \omega_i(\xi) := |v_i|^2 - 2\xi_i + 4 \sum_j \xi_j, \quad \Omega_k(\xi) = |k|^2 + 4 \sum_i \xi_i.$$

By (\cdot, \cdot) we denote the real scalar product. Finally the quadratic form is

$$(36) \quad Q_M(w) = 4 \sum_{\substack{1 \leq i \neq j \leq m \\ h, k \in S^c}}^* \sqrt{\xi_i \xi_j} e^{i(x_i - x_j)} z_h \bar{z}_k + \\ 2 \sum_{\substack{1 \leq i < j \leq m \\ h, k \in S^c}}^{**} \sqrt{\xi_i \xi_j} e^{-i(x_i + x_j)} z_h z_k + 2 \sum_{\substack{1 \leq i < j \leq m \\ h, k \in S^c}}^{**} \sqrt{\xi_i \xi_j} e^{i(x_i + x_j)} \bar{z}_h \bar{z}_k.$$

Remark 4.22. The separation of the Hamiltonian in two parts is justified by the following two facts.

- $Q_M(w)$ commutes with the variables x .
- D has a natural *diagonal form*. It can be useful to identify the term $\sum_k \Omega_k(\xi) |z_k|^2$ with its symplectic representation $\Omega = \text{diag}(\{\Omega_k\}_{k \in S^c}, -\{\Omega_k\}_{k \in S^c})$

Remark 4.23. We have grouped in the perturbation P all the terms of $H_0^{>2}$, this is just a convention since such terms are explicit. In particular H_0 contains a quadratic part in y of the form $(y, Ay)/2$ where A is defined in Theorem 1.

Lemma 4.24. *Fixing $\lambda = r^\alpha / \max(|v_i|^2)$ the perturbation P satisfies the following bounds:*

$$\begin{aligned} \|\Pi_{F^0} P\|_{s,r}^\lambda &\leq Cr^{5\alpha}, \quad \|\Pi_{F^{1,0}} P\|_{s,r}^\lambda \leq Cr^{2+4\alpha}, \quad \|\Pi_{F^{0,1}} P\|_{s,r}^\lambda \leq Cr^{1+5\alpha/2}, \\ \|\Pi_{F^{0,2}} P\|_{s,r}^\lambda &\leq Cr^{2+2\alpha}, \quad \|\Pi_{V^{>2}} P\|_{s,r}^\lambda \leq Cr^{3+\alpha/2}. \end{aligned}$$

Proof. All the bounds are purely dimensional, notice only that we have used constrain 2 to impose that the first non-zero contribution to F^0 is from a polynomial of degree 10 and not 6. \square

To further simplify the Hamiltonian we assume:

Constraint 3. Given any four different elements h_1, h_2, h_3, h_4 in S one has

- $h_1 \pm h_2 \neq h_3 \pm h_4$.
- $(h_1 + h_2 - h_3 - h_4, h_3 - h_4) \neq 0$.

As for the the matrix $M(x)$ we have

Lemma 4.25. *Given $h, k \in S^c$ there exist at most one couple $v_i \neq v_j \in S$ such that one and only one of the next two properties holds:*

- (1) $h - k = v_j - v_i$, $|h|^2 - |k|^2 = |v_j|^2 - |v_i|^2$.
In this case one has $M_{(h,\sigma),(k,\sigma)} = \bar{M}_{(k,\sigma),(h,\sigma)} = 4\sigma \sqrt{\xi_i \xi_j} e^{i\sigma(x_i - x_j)}$ and $M_{(h,\sigma),(h,-\sigma)} = 0$.
- (2) $i < j$, $h + k = v_i + v_j$, $|h|^2 + |k|^2 = |v_i|^2 + |v_j|^2$.
In this case one has $M_{(h,-\sigma),(k,\sigma)} = -\bar{M}_{(k,\sigma),(h,-\sigma)} = 4\sigma \sqrt{\xi_i \xi_j} e^{i\sigma(x_i + x_j)}$ and $M_{(h,\sigma),(k,\sigma)} = 0$.
- (3) If there exists no couple $v_i \neq v_j \in S$ satisfying either (1) or (2) then $M_{(h,\pm),(k,\pm)} = 0$ (this includes naturally the case $h = k$).

Proof. With the notations of Formula (36) constraint 4.6 now means that \sum^* denotes the constraint given in Item (1) and \sum^{**} , the constraint given in Item (2).

By Formula 16 we have

$$\{4\sqrt{\xi_i \xi_j} e^{i(x_i - x_j)} z_h \bar{z}_k, z_k\} = 4i\sqrt{\xi_i \xi_j} e^{i(x_i - x_j)} z_h.$$

The terms $2\sqrt{\xi_i \xi_j} e^{-i(x_i + x_j)} z_h z_k$ repeat twice and

$$\{\sqrt{\xi_i \xi_j} e^{-i(x_i + x_j)} z_h z_k, \bar{z}_k\} = -i\sqrt{\xi_i \xi_j} e^{-i(x_i + x_j)} z_h$$

while

$$\{\sqrt{\xi_i \xi_j} e^{i(x_i + x_j)} \bar{z}_h \bar{z}_k, z_k\} = i\sqrt{\xi_i \xi_j} e^{i(x_i + x_j)} \bar{z}_h.$$

Item (3) is trivial by the definitions. The fact that the two different conditions cannot hold contemporarily with the same v_i, v_j is again trivial. We only need to prove that (1) and (2) cannot hold for two different pairs. Suppose now that item (1) holds for some h, k and for two different pairs $\{v_i, v_j\}, \{v_l, v_m\}$ then $v_i - v_j = v_l - v_m$ contrary to hypothesis 3. The same if item (2) holds for some h, k and for two different pairs $\{v_i, v_j\}, \{v_l, v_m\}$.

Suppose now that for some h, k item (1) holds with $\{v_i, v_j\}$ and item (2) holds with $\{v_l, v_m\}$, then by substituting we contradict the second item of 3. \square

5. GRAPH REPRESENTATION

It will be convenient to associate to $ad(N)$ or equivalently to the matrix M (associated to (36) through Formula (29)) two graphs Λ_S, Γ_S encoding the information of its non-zero off diagonal entries. In fact in $ad(N)$ the part associated to D is diagonal.

The two graphs arise from the following complementary points of view.

5.1. Geometric graph Γ_S . The geometric point of view is to consider the action of $ad(N)$ on the space $V^{(0,1)}$. In fact by a simple inspection from Lemma 4.25 we have

Proposition 5.2. *The operator $ad(N)$ preserves the subspace $V_f^{(0,1)}$ of $V^{(0,1)}$ of finite linear combinations of the elements z_k, \bar{z}_k , $k \in S^c$ with coefficients in the algebra A of finite Fourier series in the variables x .*

We shall refer to z_k, \bar{z}_k , $k \in S^c$ as *the geometric basis* (of $V_f^{(0,1)}$ as a free module over the algebra A).²

For the geometric graph we shall initially forget the difference between z_k, \bar{z}_k and remember only the vector k .

Definition 5.3. The graph Γ_S has vertices \mathbb{Z}^n and edges corresponding to non zero entries of M .

In order to keep track of complex conjugation it is convenient to *decorate* the edges of the graph with two colors, black and red according to the rules that we presently explain, and a *marking or label*.

1. If item (1) in 4.25 holds we connect h, k with a black edge oriented from k to h and labeled by $v_j - v_i$;
2. If item (2) in 4.25 holds we connect h, k with a red non-oriented edge labeled by $v_j + v_i$;
3. If item (3) holds we do not connect h and k .

These rules are purely geometric and can in fact be applied to all vectors in \mathbb{R}^n or even \mathbb{C}^n and not just integral vectors. They define thus a *colored geometric graph* with vertices points in space and edges given by the previous rules. To be specific:

Definition 5.4. Given two vectors v_h, v_k we set $S_{h,k} = S_{k,h}$ to be the sphere having as one of its diameters v_h, v_k . It has the equation

$$(37) \quad (x - v_h, x - v_k) = 0, \quad \text{or} \quad (x, v_h + v_k) = (v_h, v_k) + (x)^2.$$

We also define the hyperplane

$$H_{h,k} := \{x \mid (x, v_k - v_h) = (v_k, v_k - v_h)\}.$$

Remark 5.5. The hyperplanes $H_{h,k}$ and $H_{k,h}$ are parallel and

$$H_{h,k} = v_k - v_h + H_{k,h}.$$

Observe that by definition $v_h \in H_{h,k}$ but if $v_h \in H_{i,j}$, $i \neq h$ we must have $(v_h, v_j - v_i) = (v_i, v_j - v_i)$. By Constraint 1. this is not satisfied, if $i \neq h$.

In other words, if $i \neq h$ then $v_i \notin H_{h,k}$.

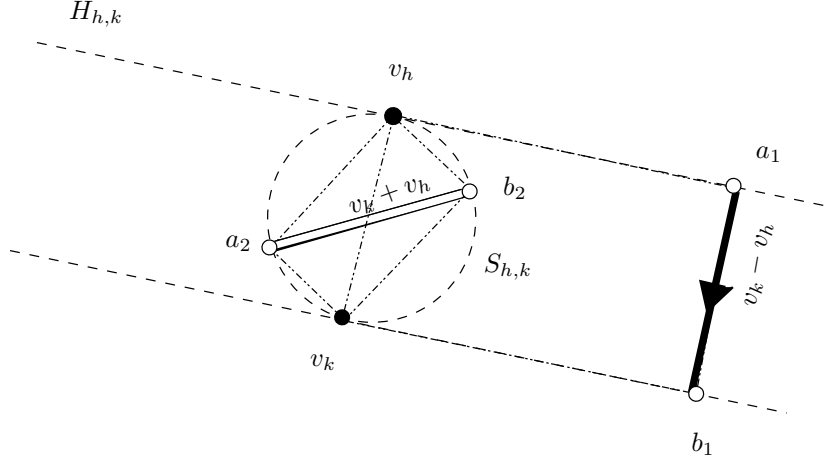


FIGURE 1. The plane $H_{h,k}$ and the sphere $S_{h,k}$. The points a_1, b_1, v_k, v_h form the vertices of a rectangle. Same for the points a_2, v_h, b_2, v_k

We construct now the *colored graph* Γ_S with vertices in \mathbb{R}^n . Consider two points $a, b \in \mathbb{R}^n$ (sometimes even in \mathbb{C}^n).

If there exists a pair h, k , $h \neq k$ so that $a \in H_{h,k}$, $b = a + v_k - v_h \in H_{k,h}$ we join the two points by a *black edge* oriented from a to b and marked $v_k - v_h$. See the points $\{a_1, b_1\}$ in Figure 1. In other words:

$$(38) \quad \{a, b\} \left| \begin{cases} |b_1|^2 - |a|^2 = |v_k|^2 - |v_h|^2 \\ b - a = v_k - v_h \end{cases} \right.$$

or equivalently $(a, v_h - v_k) = (v_h, v_h - v_k)$.

If there exists a pair h, k , $h \neq k$ so that $a \in S_{h,k}$, $b = -a + v_k + v_h \in S_{k,h} = S_{h,k}$ we join the two points by a *red edge* marked by $v_h + v_k$. See the points $\{a_2, b_2\}$ in Figure 1 (we represent red edges by a double line).

$$(39) \quad \{a, b\} \left| \begin{cases} |a|^2 + |b|^2 = |v_h|^2 + |v_k|^2 \\ a + b = v_h + v_k \end{cases} \right.$$

In other words $(a - v_h, a - v_k) = (b - v_h, b - v_k) = 0$.

The points a, v_h, b, v_k form the vertices of a rectangle. In other words,

Lemma 5.6. a, b are opposite points in the sphere $S_{h,k}$ having as one of its diameters v_h, v_k .

Remark 5.7. For each pair h, k there are finitely many points in \mathbb{Z}^n , in the sphere $S_{h,k}$. Therefore there are only finitely many red edges with integral vertices.

Definition 5.8. We construct the graph Γ_S with vertices all the points of \mathbb{R}^n and edges the black and red edges described.

²When the coefficients in a basis are in an algebra and not a field it is customary to use the word *module* and not vector space.

We want to understand the connected components of the graph Γ_S .

By Constraint 1 if $i \neq h, i \neq k$ then $v_i \notin S_{h,k}$. We immediately have.

Lemma 5.9. *The vectors v_1, \dots, v_m are a component of the graph Γ_S . In this component every two vertices are joined by a red and by a black edge.*

Proof. If v_i is joined to another vector u by a black edge of type $H_{h,k}$ we must have by constraint 1 that $h = i$ and $u = v_k$. Similarly for a red edge of type $S_{h,k}$ using constraint 1 we have $i = h$ or $i = k$. \square

Definition 5.10. The component v_1, \dots, v_m is called the *special component* of the graph Γ_S .

5.11. The combinatorial graph Λ_S . Consider the space $F^{(0,1)}$ with its basis over \mathbb{C} given by Remark 4.16.

This basis is really indexed by $\mathbb{Z}^m \times \mathbb{Z}/(2)$, that is an integral vector plus a sign, we shall refer to it as *the frequency basis*.

From Lemma 4.25 the linear operator $ad(N)$ has the property that it transforms every element of this basis into a finite linear combination of the same basis.

Definition 5.12. The graph Λ_S has vertices $\mathbb{Z}^m \times \mathbb{Z}/(2)$ and edges corresponding to non zero entries of $ad(N)$.

5.13. Abstract colored marked graphs. It will be useful to also use completely abstract graphs defined as follows

Definition 5.14. A *abstract colored marked graph* or \mathcal{M} -*graph* for short, is

- A connected graph Γ (without repeated edges).
- A color *red* or *black* on each edge, displayed

$$a \xrightarrow{\text{black}} b \qquad c \xrightarrow{\text{red}} d$$

- A marking (i, j) , $1 \leq i \leq m, 1 \leq j \leq m, i \neq j$ on each oriented edge with the convention that the opposite orientation corresponds to the exchanged marking (j, i) .

A *geometric realization* of the graph Γ is a graph isomorphism with a connected component of Γ_S such that each black edge of Γ marked (i, j) corresponds to a black edge of A marked $v_j - v_i$. In the same way to each red edge of Γ marked (i, j) corresponds to a red edge of A marked $v_i + v_j$.

5.15. Summary of results. In term of the frequency basis, denote by e_i the basis of \mathbb{Z}^m and consider the map

$$(40) \qquad \pi : \mathbb{Z}^m \rightarrow \mathbb{Z}^n, \qquad \pi(\nu_1, \dots, \nu_m) := \sum_{i=1}^m \nu_i v_i.$$

If $k = -\pi(\mu)$, the vector $e^{i\mu \cdot x} z_k$ lies in $F^{(0,1)}$. The vectors $z_k \in V^{(0,1)}$, $e^{i\mu \cdot x} z_k$ map under $ad(N)$ to a linear combination in which respectively:

- If h, k are connected with a black edge oriented from k to h and labeled by $v_j - v_i$:
 - $h = -\pi(\mu + e_i - e_j) = k + v_j - v_i$
 - In the geometric basis, z_h has coefficient $4\sqrt{\xi_i \xi_j} e^{i(x_i - x_j)}$ in $M(z_k)$.
 - In the frequency basis, $e^{i(\mu + e_i - e_j) \cdot x} z_h$ has coefficient $4i\sqrt{\xi_i \xi_j}$ in $ad(N)(e^{i(\mu \cdot x)} z_k)$.
- If h, k are connected with a red non-oriented edge labeled by $v_j + v_i$

- $h = \pi(\mu + e_i + e_j) = -k + v_i + v_j$
- In the geometric basis, \bar{z}_h has coefficient $4\sqrt{\xi_i \xi_j} e^{i(x_i + x_j)}$ in $M(z_k)$.
- In the frequency basis, $e^{i(\mu + e_i + e_j) \cdot x} \bar{z}_h$ has coefficient $4i\sqrt{\xi_i \xi_j}$ in $ad(N)(e^{i(\mu \cdot x)} z_k)$.

Remark 5.16. Under our convention on the indexing of the basis, $(\mu, +)$ corresponds to $e^{i(\mu \cdot x)} z_h$ while $(-\mu - e_i - e_j, -)$ corresponds to $e^{i(\mu + e_i + e_j) \cdot x} \bar{z}_h$. In the frequency graph Λ_S therefore, the elements $\mu, \mu + e_i - e_j$ are joined by an edge in case i). In case ii) if $\nu = -\mu - e_i - e_j$, that is if $\mu + \nu + e_i + e_j = 0$, we have that $(\mu, +), (\nu, -)$ are connected by an edge marked $e_i + e_j$.

We will show that, provided we choose the vectors v_i *generically*, we shall have several essential properties for these graphs which we will need in order to and prove the reducibility Theorem 1, and study the Homological equation.

The *generic assumption* will be expressed by the fact that the coordinates of the vectors v_i do not satisfy some polynomial equation (a product of several equations which will be constructed in the course of the proof). That is, we think of $(v_1, \dots, v_m) \in \mathbb{R}^{nm}$ and will impose that this point does not lie in a certain algebraic hypersurface whose equation will be at least implicitly given (in term of certain graphs). The precise statements are contained in §9.1.

We now discuss the properties deduced from the generic assumption.

The first relates the two graphs Λ_S, Γ_S . In fact, take a frequency μ , and let \mathcal{A} be the associated component in Λ_S . Set $k = -\pi(\mu)$ and A be the associated component in Γ_S . We shall see

Theorem 4. *The map $-\pi$ establishes a graph isomorphism between \mathcal{A} and A , compatible with the markings.*

Hence the space spanned by all transforms of $e^{i\mu \cdot x} z_k$ applying the operator $ad(M)$ has a basis extracted from the frequency basis in correspondence, under π , with the vertices of A .

Same statement for its conjugate generated by $e^{-i\mu \cdot x} \bar{z}_k$.

Remark 5.17. We shall often refer to such a subspace as a *block*, for $ad(N)$. For generic S the symplectic form restricted to this block is identically 0 and if we add a block with its conjugate we have a non degenerate symplectic space decomposed as sum of two $ad(N)$ -stable Lagrangian subspaces.

This is thus a block for $ad(M)$ for the graph Λ_S , and all other blocks in the set of elements $\nu \mid -\pi(\nu) \in A$ are obtained from this block by multiplying with all the elements ν such that $\pi(\nu) = 0$.

The entire space $F^{(0,1)}$ therefore decomposes into free submodules under the algebra C of finite Fourier series in $e^{i\nu \cdot x} \mid \pi(\nu) = 0$ corresponding to all the geometric blocks in Γ_S . For each such block A a basis of the corresponding space over C is obtained as follows. We choose a specific element $r \in A$, a *root* and then a specific μ with $\pi(\mu) = -r$. From μ , and applying $ad(N)$, we construct the basis for the submodule in correspondence with the vertices of A .

Let us summarize the most important properties that we shall prove in §7 and §8.

Theorem 5. *i) All connected components of the graph Γ_S have as vertices points which are affinely independent hence at most $n + 1$ vertices.
ii) There are finitely many components containing red edges.*

iii) The connected components of Γ_S consisting only of black edges are divided into a finite number of families.

Each family is indexed by an abstract marked graph with $k \leq n$ edges, and it depends on the elements of a $n - k$ dimensional sub-lattice.

Given a connected component A with $k + 1$ vertices ($k \leq n$) of Γ_S we fix a vertex $x(A) \in A$ which we call the root. Then:

Corollary 5.18. For every other vertex x_a (with $a = 1, \dots, k$) one has two functions on A :

$$(41) \quad \sigma(a) = \pm 1, \quad L(a) = \sum A_a^{(i)} e_i, \quad A_a^{(i)} \in \mathbb{Z};$$

$$|L(a)| < n, \quad \sum_i A_a^{(i)} = 1 - \sigma(a),$$

such that $\sigma(a)$ is 1 if the path from $x(A)$ to x_a has an even number of red edges, -1 otherwise. We have from Formulas (71) and (72)

$$(42) \quad x_a = \sigma(a)x(A) + \pi(L(a)) = \sigma(a)x(A) + \sum A_a^{(i)} v_i,$$

$$|x_a|^2 = \sigma(a)|x(A)|^2 + \sum A_a^{(i)} |v_i|^2.$$

The matrix M is block diagonal with two blocks (denoted by A, \pm) in correspondence with each connected components A of Γ_S .

The matrix M restricted to $A, +$ is denoted by $M_{A,+}$ and given by

$$M_{a,b} := M_{(x(a), \sigma(a)), (x(b), \sigma(b))} = 0$$

if (a, b) is not an edge (in particular it is zero on the diagonal).

$$M_{a,b} := M_{(x(a), \sigma(a)), (x(b), \sigma(b))} = 4\sigma(b)\sqrt{\xi_i \xi_j} e^{i(\sigma(b)L(b) - \sigma(a)L(a)) \cdot x}$$

if $(a, b) = e$ is an edge marked (i, j) .

$M_{A,-} = -\overline{M_{A,+}}$ is minus the conjugate of $M_{A,+}$.

Proof. We fix a root x on A and a sign $\sigma(x) = +$. We associate to each vertex in A a corresponding vertex in \mathcal{A} , by Theorem 4. This associates to each x_a a sign $\sigma(a)$. We use the relations (38) and (39) to compute the $A_i^{(a)}$ by choosing a path from x to each vertex x_a . Since π is a graph isomorphism, compatible with the two markings, it defines $L(a)$ uniquely. Notice that if say $\sigma(b) = +$ then $\sigma(b)L(b) - \sigma(a)L(a) = e_j - e_i$ if the edge is black and $\sigma(b)L(b) - \sigma(a)L(a) = e_j + e_i$ otherwise. To prove the second statement we implement the matrix rules of Lemma 4.25. \square

6. PROOF OF THEOREMS 1, 2 AND 3

6.1. Theorem 1. We will prove Theorem 1 by exploiting the block structure discussed in Corollary 5.18. For all $k \in S^c$ set $x(k) := x(A)$ to be the root of the component A of Γ_S to which it belongs. Set $L(k) = 0$ if $k = x(A)$ is the root (this includes the connected components made of one point). Otherwise $k = x_b$ for some index $b = b(k)$. We then set $L(k) := L(b)$ and $\sigma(k) = \sigma_b$ (cf. 5.18). Theorem 1 is contained in the following, more precise, propositions:

Proposition 6.2. i) The equations

$$(43) \quad z_k = e^{iL(k) \cdot x} z'_k, \quad y = y' - \sum_{k \in S^c} L(k) |z'_k|^2, \quad x = x'.$$

define a symplectic change of variables $D(s, r/2) \rightarrow D(s, r)$, which preserves the spaces $V^{i,j}$.

We denote by $X = \text{diag}(\{e^{iL(k).x}\}_{k \in S^c}, \{e^{-iL(k).x}\}_{k \in S^c})$, the change of variables on w and $\Omega' = \text{diag}(\{\Omega_k - (\omega, L(k))\}_{k \in S^c}, -\{\Omega_k - (\omega, L(k))\}_{k \in S^c})$.

ii) The Hamiltonian H in the new variables is

$$N + (y', Ay') + P^{(3)} + P^{(6)}$$

where N in the new variables is

$$(44) \quad N := (\omega(\xi), y') + Q_{M'}(w'), \quad M' = \Omega' + XMX^{-1},$$

and the terms $P^{(3)}, P^{(6)}$ satisfy the bounds of Theorem 1, v).

Proof. i) Since

$$\sup_{D(s, r/2)} |w'|_{a,p} \leq e^{Cs} |w|_{a,p} \leq e^{Cs} r/2 \leq r$$

for s small enough the transformation is well defined from $D(s, r/2)$ to $D(s, r)$. It is symplectic because:

$$\begin{aligned} dy \wedge dx + idz \wedge d\bar{z} &= dy' \wedge dx' - \sum_k L(k) d(|z_k|^2) \wedge dx' + \\ idz' \wedge d\bar{z}' - \sum_k L(k).dx'(z'_k \wedge d\bar{z}'_k - \bar{z}'_k \wedge dz'_k) &= dy' \wedge dx' + idz' \wedge d\bar{z}'. \end{aligned}$$

Finally it preserves the spaces $F^{i,j}$ since it is linear in the variables w .

ii) We substitute the new variables in the Hamiltonian and use the relation $JX = X^{-1}J$, i.e. the fact that X is symplectic. The bounds follow from Lemma 4.24, notice that we have put all the terms in $H_0^{>2}$ which are not quadratic in y in the perturbation $P^{(6)}$ where they contribute to the terms of weight ≥ 3 with a term of order $r^{4+\alpha/2}$ which is negligible with respect to $P^{(3)}$. \square

From an algebraic point of view, we have performed a diagonal change of coordinates using the matrix X on the free module $V_f^{(0,1)}$. Recall that this is the space of finite linear combinations of the element z_k, \bar{z}_k , $k \in S^c$ with coefficients in the algebra A of finite Fourier series in the variables x .

It is clear that the block structure is still preserved, the main fact is now that the matrix has constant coefficients.

Proposition 6.3. i) M' has constant coefficients and is block diagonal with the same block structure as M . On a block $(A, +)$ with root $x(A)$,

$$(45) \quad M'_A = (|x(A)|^2 + 4 \sum_j \xi_j)I + 2C_A,$$

where C_A has the following entries in the vertices a, b .

$$(46) \quad C_A(a, b) = \begin{cases} 0 & \text{if } M_{a,b} = 0, \quad a \neq b \\ 4\sigma(b)\sqrt{\xi_i\xi_j} & \text{if } (a, b) = e \text{ is an edge marked } (i, j). \\ \sigma(a)(\xi, L(a)) & \text{if } a = b. \end{cases}$$

ii) C_A is self-adjoint if A does not contain red edges. If A contains red edges there is a diagonal matrix Σ with entries ± 1 such that ΣC_A is self adjoint.

iii) The matrix C_A depends only from the abstract \mathcal{M} -graph corresponding to A and hence is chosen from a finite list.

Proof. i) By definition $M' = \Omega' + XM X^{-1}$ so let us analyze separately these two operators. Since X is diagonal the matrix $XM X^{-1}$ is 0 on the diagonal.

The off diagonal entry associated to the indices a, b of two vertices in the given block is 0 unless they are joined by an edge marked (i, j) . Then

$$(47) \quad (XM X^{-1})_{a,b} = \sigma(b) 4 \sqrt{\xi_i \xi_j} e^{\sigma(a) i L(a) \cdot x} e^{-\sigma(b) i L(b) \cdot x} e^{i(\sigma(b) L(b) - \sigma(a) L(a)) \cdot x} = 4 \sigma(b) \sqrt{\xi_i \xi_j}$$

On the diagonal of M' we have the contribution of Ω' . Applying Formulas (42) and (35), since by (41) we have $\sum_i A_a^{(i)} = 1 - \sigma(a)$, we get:

$$(48) \quad \begin{aligned} \Omega'_a &= \sigma(a) (|k(a)|^2 + 4 \sum_j \xi_j - (\omega, L(a))) = \\ &= |x(A)|^2 + \sigma(a) \sum A_a^{(i)} |v_i|^2 + \sigma(a) (4 \sum_j \xi_j - (\omega, L(a))) = \\ &= |x(A)|^2 + \sigma(a) \sum A_a^{(i)} |v_i|^2 + \sigma(a) (4 \sum_j \xi_j - \sum_i (|v_i|^2 - 2\xi_i + 4 \sum_j \xi_j) A_a^{(i)}) = \\ &= |x(A)|^2 + \sigma(a) (4 \sum_j \xi_j - \sum_i (-2\xi_i + 4 \sum_j \xi_j) A_a^{(i)}) = \\ &= |x(A)|^2 + \sigma(a) [2(\xi, L(a)) + 4 \sum_j \xi_j - 4 \sum_j \xi_j (1 - \sigma(a))] = \\ &= |x(A)|^2 + \sigma(a) 2(\xi, L(a)) + 4 \sum_j \xi_j. \end{aligned}$$

We then define $2C_A$ to have off-diagonal entries given by (47) and on the diagonal $2\sigma(a)(\xi, L(a))$.

ii) Is immediate.

iii) In the matrix C_A the off-diagonal entries depend only on which pairs a, b are connected by a marked edge, which depends only on the abstract \mathcal{M} -graph. In the same way the diagonal entries depend on $L(a)$ which depends only on the path from the root to a (again this depends only on the abstract marked graph). Finally there are only a finite number of abstract marked graphs with $k \leq n + 1$ vertices. □

Remark 6.4. Notice that in the new variables the term H_0 is independent of x .

6.5. Combinatorial blocks. We have described the matrix M' in the basis z'_k . In particular, for a given geometric block A we have chosen the block in $V^{(0,1)}$ generated by the element $z_{x(A)} = z'_{x(A)}$. We have to understand in this formalism the elements in $F^{(0,1)}$ —that is momentum conservation and parity in the new variables— and then compute the matrices of the operator $ad(N)$ on each combinatorial block of the graph Λ_S (cf. Definition 5.12).

Lemma 6.6. *Momentum conservation (26) and parity in the new variables give the constrain:*

$$(49) \quad e^{i\nu \cdot x} y^i \prod_k (z'_k)^{\alpha_k} (\bar{z}'_k)^{\beta_k} \in F^{i,j} \rightarrow \sum_i \nu_i v_i + \sum_k \sigma(k) (\alpha_k - \beta_k) x(k) = 0,$$

$$\sum_i \nu_i + \sum_k (\alpha_k + \beta_k) = \text{even}.$$

In particular $e^{i\sigma(k)\nu \cdot x} z'_k \in F^{0,1}$ if and only if $x(k) = -\pi(\nu)$ and $\sum_i \nu_i$ is odd.

Proof. The momentum conservation in the variables w' can be derived directly from (26) by substitution:

$$e^{i\nu \cdot x} \prod_k (z'_k)^\alpha (\bar{z}'_k)^\beta = e^{i(\nu \cdot x - \sum_k (\alpha_k - \beta_k) L(k) \cdot x)} \prod_k (z_k)^\alpha (\bar{z}_k)^\beta$$

then, if $L(k) = \sum_i A(k)^{(i)} e_i$ (cf. Corollary 5.18), momentum conservation reads

$$(50) \quad \sum_i \nu_i v_i - \sum_i \sum_k (\alpha_k - \beta_k) A(k)^{(i)} v_i + \sum_k (\alpha_k - \beta_k) k.$$

Recalling that, by Corollary 5.18 with $k = x_a$, $k - \pi(L(k)) = \sigma(k)x(k)$ one obtains formula (49). The parity condition is that $\sum_i (\nu_i - \sum_k (\alpha_k - \beta_k) A(k)^{(i)}) + \sum_k (\alpha_k + \beta_k)$ is even. In Corollary 5.18 we have seen that $\sum_i A(k)^{(i)} = 0, 2$ and the parity follows. \square

We have described the matrix M' on geometric blocks. From this description we can deduce a description of the operator $ad(N)$ on the combinatorial blocks, in particular on $F^{(0,1)}$. A combinatorial block $(\mathcal{A}, +)$ over a given geometric block A is generated by an arbitrary lift $e^{i\nu \cdot x} z_{x(A)}$ of the root $x(A)$ in $F^{(0,1)}$ (i.e. by choosing a ν such that $x(A) = -\pi(\nu)$). By Lemma 6.6, the operator $(\omega(\xi), y')$ contributes to $ad(N)_{\mathcal{A}}$ the scalar matrix $-2i(\xi, \nu)I$.

Definition 6.7. Given a combinatorial block \mathcal{A} generated by the lift $e^{i\nu \cdot x} z_{x(A)}$ we define $C_{\mathcal{A}} = C_A - (\xi, \nu)I$. For the blocks $\mathcal{A}, -$ we apply sign change.

Proposition 6.8. *let A be a geometric block with root $x(A)$. On the combinatorial block $(\mathcal{A}, +)$ generated by an element $e^{i\nu \cdot x} z_{x(A)}$ the operator $ad(N)$ has matrix:*

$$(51) \quad -i ad(N)_{\mathcal{A}} = \left(|x(A)|^2 + \sum_i \nu_i |v_i|^2 + 4 \left(\sum_i \nu_i + 1 \right) \sum_j \xi_j \right) I + 2C_{\mathcal{A}}.$$

Remark 6.9. Notice that, for any given block A , the two combinatorial blocks $\mathcal{A}, +$ and $\mathcal{A}, -$ form two Lagrangian subspaces of a non degenerate symplectic space and the full space $F^{(0,1)}$ is the direct sum of these subspaces.

6.10. Theorem 2. The core of Theorem 2 is to put the normal form N in a canonical form through a linear change of variables in the w' which depends smoothly on the parameters ξ .

Let Σ be a diagonal matrix with p entries equal to 1 and $q = n - p$ entries equal to -1 . If $p > 0, q > 0$ the matrix Σ defines an indefinite scalar product preserved by a non-compact form of the orthogonal group usually denoted $O(p, q)$. If C is an $n \times n$ matrix such that ΣC is symmetric then it can be brought into a suitable canonical Jordan form by conjugation with elements of $O(p, q)$. Is furthermore C is semisimple then this normal form consists in decomposing the space \mathbb{R}^n into orthogonal subspaces stable under C and irreducible under C . These have either dimension 1 and correspond to the real eigenvalues or dimension 2 and correspond to the pairs of complex conjugate eigenvalues (one can also see that we have at most $\min(p, q)$ of such 2 dimensional subspaces. In the case of our blocks containing red edges we have chosen a basis where the symplectic products are of the form $\pm i$ associated to such a sign matrix Σ . For such a symplectic

block sum of $A, +$ and $A, -$ a matrix of the form $\begin{vmatrix} U & 0 \\ 0 & U \end{vmatrix}$ with $U \in O(p, q)$ is symplectic and brings $\begin{vmatrix} C & 0 \\ 0 & -C \end{vmatrix}$ in a *diagonal* normal form (of course some diagonal blocks are 2×2 and are matrices associated to complex numbers).

We need to study matrices which depend algebraically on parameters, we have the following proposition:

Proposition 6.11. *If $C(\xi)$ depends algebraically on parameters $\xi \in \mathbb{R}^m$, one can define globally and algebraically its eigenvalues provided we cut some real semialgebraic hypersurface (in an arbitrary way) so that the complement is simply-connected.*

If $C(\xi)$ is semisimple on an open set, one can define globally and algebraically the change of coordinates which brings $C(\xi)$ in the given diagonal form provided we cut some real semialgebraic hypersurface (in an arbitrary way) so that the complement is simply-connected.

This is a fairly standard fact and we leave out the proof.

Proof of Theorem 2. i) By formula (45), on the symplectic block (A, \pm) the quadratic Hamiltonian N is given by the matrix

$$(52) \quad \mathcal{M}_A = (|x(A)|^2 + 4 \sum_j \xi_j) \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix} + 2 \begin{vmatrix} C_A & 0 \\ 0 & -C_A \end{vmatrix},$$

where $x(A)$ is the root of A while C_A depends only on the combinatorial block \mathcal{A} of which A is a realization. and let $\lambda_1(\xi), \dots, \lambda_k(\xi)$ be its eigenvalues. We denote by $\tilde{\Omega}_i = \tilde{\Omega}_i(\xi)$ the corresponding eigenvalues of \mathcal{M}_A^+ thus

$$(53) \quad \tilde{\Omega}_i(\xi) = |x(A)|^2 + 4 \sum_j \xi_j + \lambda_i(\xi).$$

Suppose that \mathcal{M}_A is semisimple, by Proposition 6.11 \mathcal{M}_A can be conjugated by a real invertible matrix U in $D_{s,k}$. Moreover for ξ outside a semialgebraic set the matrix $U(\xi)$ is algebraic in the ξ . We proceed as described above for all the \mathcal{M}_A which are self-adjoint, i.e. for all A that do not contain red edges. We apply $w' \rightarrow w^{\text{fin}} = U_A w'$ so that $Q_{\mathcal{M}_A}(w') = Q_{D_A}(w^{\text{fin}})$ where

$$D_A = \begin{pmatrix} \mathcal{D}_A & 0 \\ 0 & -\mathcal{D}_A \end{pmatrix}, \quad \mathcal{D}_A = \text{diag}(\tilde{\Omega}_{k(a)})$$

\mathcal{D}_A is the diagonal matrix of the (real) eigenvalues of \mathcal{M}_A . Now the total transformation U is block diagonal. Since $|k(a)|^2 \leq |x(a)|^2 + C$ for some uniform C (depending only on the size $\max_{v_i \in S} |v_i|^2$) we have that

$$|w^{\text{fin}}|_{a,p} \leq C|w_+|_{a,p}.$$

Item i) follows by setting R' to be the matrix of $\sum_{k,h \in \mathcal{H}} (w_h^{\text{fin}}, M'_{h,k} J w_k^{\text{fin}})$.

ii) The $\tilde{\Omega}_k$ are the eigenvalues of the matrices \mathcal{M}_A of Formula (52) as A varies of the connected components of Γ_S . Since by Proposition 6.2 iv) C_A depends only on the abstract M-graph, there is only a finite number of such matrices. Moreover the entries of these matrices are homogeneous of degree one in ξ . We apply Proposition 6.11; we cut away a semialgebraic set \mathfrak{A} such that the complement is simply connected. In $A_\alpha \setminus \mathfrak{A}$ we obtain a list of eigenvalues $\lambda^{(i)}(\xi)$ and a list of matrices (with determinant equal to one)

$U_i(\xi)$ which diagonalize the C_A which are semisimple (this set contains at least all the C_A which do not contain red edges). These are all algebraic functions.

Let \mathfrak{A}_ϵ be a neighborhood of \mathfrak{A} of radius ϵ then in $\tilde{A}_\alpha := A_\alpha \setminus \mathfrak{A}_\epsilon$ the $\lambda^{(j)}$ are analytic functions in x , the bounds (14) follow by Cauchy estimates by setting $\epsilon = r^\alpha/4$.

iv) Follows directly from Lemma 4.24 since the linear change of variables $(x, y, z) \rightarrow (x', y', z^{fin})$ does not modify the bounds. Indeed the only point is to prove that the Lipschitz bounds are unchanged, this follows by the estimates of item *iii*) and the fact that in \tilde{A}_α also the eigenspaces are analytic in the parameters ξ . \square

Remark 6.12. Suppose that C_A for some A is semisimple for all $\xi \in A_\alpha$. If the eigenvalues of C_A are real then one can proceed as above. In the case of hyperbolic blocks (with eigenvalues $a \pm ib$) we obtain the Hamiltonian:

$$-a(z_2 z_1 + \bar{z}_2 \bar{z}_1) - b \frac{\bar{z}_2^2 + z_2^2}{2} = -\frac{1}{2} w \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} J w^t, \quad D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

we have proved Remark 3.3.

In general however C_A may not be semisimple and thus have Jordan blocks of size > 1 . A normal form classification of this quadratic Hamiltonian is feasible, see [1], but intricate and not particularly useful.

6.13. The homological equation. The homological equation consists in the analysis of the range and kernel of the operator $ad(N)$ on $F^{\leq 2}$. Using the block diagonal form (32) this analysis can be split into 4 different equations. The crucial ones are the last two which for historical (and confusing) reasons are called the first and second Melnikov equations.

In order to study the Homological equation it is not really necessary to perform the reduction of Proposition 6.2, since we have seen that the change of variables leaves it essentially unchanged. In the same way it is not necessary to diagonalize the Hamiltonian, since this just conjugates the operator $ad(N)$. The main point in the homological equation is to study the Kernel of $ad(N)$ on the real subspace of $F^{\leq 2}$, and to show that the kernel coincides with the subspace F^{int} :

$$F^{int} := \{F \in F^{\leq 2} \mid F(\lambda, W) = c + (\lambda(\xi), y) + Q_{W(\xi)}(w)\}$$

where c is a constant, $\lambda \in \mathbb{R}^m$ and W is a block-diagonal matrix with the same blocks as M' and (block by block) simultaneously diagonalizable with M' .

Lemma 6.14. *For any given ξ the condition $\ker(ad(N)) = F^{int}$ is equivalent to the fact that the non-trivial Melnikov resonances (5) do not occur.*

Proof. By Lemma 4.19 $ad(N)$ restricted to $F^{0,0}$ and $F^{1,0}$ is the operator $\omega(\xi) \cdot \partial_x$ which has eigenvalues $(\omega(\xi), \nu)$.

By Lemma 4.19 $ad(N)$ restricted to $F^{0,1}$ has eigenvalues $i((\omega(\xi), \nu) \pm \tilde{\Omega}_{k(a)})$ and hence is invertible if they are non-zero.

For $F^{0,2}$ we use Lemma 4.20: for each combinatorial block consider a basis of eigenvectors a_i^A for $ad(N)$ on \mathcal{A} , + denote by b_i^A the dual basis in \mathcal{A} , -. A basis for $F^{0,2}$ formed by eigenvectors of $ad(N)$ is obtained by making all the possible products of two eigenvectors in the basis. The corresponding eigenvalue is the sum of the two eigenvalues.

Thus $ad(N)$ restricted to $F^{0,2}$ is invertible on an eigenvector (say for example $a_i^A b_j^B$) if the corresponding Melnikov resonance does not occur. Moreover the trivial Melnikov resonances $a_i^A b_i^A$ span $F^{int} \otimes \mathbb{C} \cap F^{0,2}$. \square

Proof of Theorem 3 item i). Given a block A with root $x(A)$, let $(\mathcal{A}, +)$ be a corresponding combinatorial block. By the previous Lemma we need to show that $ad(N)_{\mathcal{A}}$ is invertible. The operator $-iad(N)_{\mathcal{A}}$ computed at $\xi = 0$ is the scalar matrix $(\omega(0), \nu) \pm |x(A)|^2$, if this number is non zero then $ad(N)_{\mathcal{A}}$ is invertible for all ξ small. If $ad(N)_{\mathcal{A}}(\xi = 0) = 0$ then we study $-\frac{i}{2}ad(N)_{\mathcal{A}}$ using (51). By definition $C_{\mathcal{A}}$ has off-diagonal entries $2\sqrt{\xi_i \xi_j}$, thus $\frac{i}{2}ad(N)_{\mathcal{A}}$ modulo 2 is diagonal. Moreover, since $\sum_i \nu_i$ is odd while $\sum_i A(k)^{(i)}$ is even (see the proof of the parity property), we have that $\frac{i}{2}ad(N)_{\mathcal{A}}$ modulo 2 is formally invertible hence also $ad(N)_{\mathcal{A}}$ is invertible for values of ξ outside some proper algebraic hypersurface. \square

6.15. Dimension 2 and 3. In order to prove Theorem 3 item *ii*) we need stronger conditions, which we shall verify in dimension $n = 2, 3$. In all the statements of this paragraph we assume that $n = 2, 3$.

Theorem 6. *The characteristic polynomials of all the block matrices M'_A are irreducible.*

Proof. See §11 where we do a case analysis. \square

Given a combinatorial block \mathcal{A}, \pm set $\chi_{\mathcal{A}, \pm}(t, \xi)$ to be the characteristic polynomial of the operator $ad(N)$ restricted to the given block

Corollary 6.16. *i) For all combinatorial blocks $\chi_{\mathcal{A}, \pm}(t, \xi)$ is irreducible.*

ii) For all ξ outside a zero-measure set, the eigenvalues $\tilde{\Omega}_k$ relative to the same block are all distinct. Moreover the matrix R defined in Theorem 2 i) is formed of 2×2 hyperbolic blocks as in Remark 6.12.

iii) If two different combinatorial blocks $\mathcal{A}, \pm, \mathcal{A}', \pm$ associated to two geometric graphs A, A' have a common eigenvalue of $ad(N)$ (as function of ξ) then $\chi_{\mathcal{A}, \pm}(t, \xi) = \chi_{\mathcal{A}', \pm}(t, \xi)$. In particular the roots are on the same sphere: $|x(A)|^2 = |x(A')|^2$.

Proof. The characteristic polynomial of all the blocks are obtained by translating the variable t from the polynomials of the previous Theorem so are irreducible. Since the characteristic polynomial of all the blocks is irreducible then the eigenvalues of each block are all distinct as algebraic functions. If we remove an arbitrary open set of parameters ξ which contains the zero-set of the discriminant we have for each block all distinct eigenvalues (which implies that the blocks are all diagonalizable and that the eigenvalues and eigenvectors depend smoothly on the parameters).

This follows from the fact that an irreducible polynomial is also the minimal polynomial satisfied by any algebraic function giving one of its roots, over the field of rational functions. Thus two irreducible polynomials have a root in common if and only if they are equal. This implies in particular that $|x(A)|^2 = |x(A')|^2$. \square

In order to prove Theorem 3 item *ii*), it is enough to verify the conditions of Lemma 6.14. This follows from:

Theorem 7 (Separation). *A basis for $\ker(ad(N))$ on $F^{0,2}$ is given by the elements $\alpha_i^A b_i^A$ (for two eigenvectors in conjugate blocks $\mathcal{A}, +; \mathcal{A}, -$).*

The Theorem follows from a technical Lemma. Take a combinatorial block \mathcal{A}, \pm and denote by $\chi_{\mathcal{A}, \pm}(t, \xi)$ the characteristic polynomial of the corresponding matrix $C_{\mathcal{A}}$ or $-C_{\mathcal{A}}$.

Lemma 6.17. *The block \mathcal{A} and its sign can be uniquely reconstructed by the polynomial $\chi_{\mathcal{A}, \pm}(t, \xi)$.*

Proof. See §12. □

In practice in order to prove this Lemma we shall inspect colored marked graphs with a root so that, when we embed \mathcal{A}_i in $\tilde{\Lambda}$ using Theorem 9 we have a complete list of representatives under the group of translations and sign change.

Proof of Theorem 7. This follows immediately from Corollary 6.16 and Lemma 6.17. □

Proof of Corollary 2. The only difficulty comes from the study of the second Melnikov condition i.e. the third inequality in (15). The main point is that the functions

$$(\omega(\xi), \nu) + \tilde{\Omega}_k(\xi) + \sigma \tilde{\Omega}_h(\xi) = (\omega(\xi), \nu) + L + \lambda^{(i)}(\xi) + \sigma \lambda^{(j)}(\xi),$$

are algebraic and non trivial. Moreover L is an integer while the $\lambda^{(i)}(\xi)$ are defined in (13) so that $\lambda^{(i)}(\xi) + \sigma \lambda^{(j)}(\xi)$ is a *finite* list of analytic functions, which respect the bounds (14) in the domain $A_\alpha \setminus \mathfrak{A}_\varepsilon$. Then for a given ν , we impose the second Melnikov condition for all the *infinite* choices of h, k compatible with momentum conservation by removing from A_α a *finite* number of “strips” of measure of order $\gamma(A + r^\alpha)/(1 + |\nu|^{\tau_0+1})$ (the number of “strips” is given by the number of different functions $\lambda^{(i)}(\xi) + \sigma \lambda^{(j)}(\xi)$). Finally we notice that if $A = 1$ then in order to have $(\omega(\xi), \nu) + \tilde{\Omega}_k(\xi) + \sigma \tilde{\Omega}_h(\xi) \leq 1$ we must have $|\nu| > r^{-\alpha}$. □

6.18. Dimension $n > 3$. In dimension $n > 3$ we have not been able to prove the validity of the second Melnikov condition, although we suspect that it is true, according to our Conjecture 1 in §10 and a conjectural separation Lemma. In this subsection we want to sketch how one can still apply a KAM scheme by discussing the homological equation in this more general setting. The reader should compare this discussion with the one presented in §3.6 and again with the KAM scheme developed in [3].

The first step consists in performing the canonical *Fitting decomposition* of the operator $ad(N)$, namely we decompose the space so that each block corresponds to one (and only one) of the generalized eigenvalues, as usual in our definitions we say that two eigenvalues coincide if they do so identically as functions of ξ . We define the change of variables of the Fitting decomposition on each abstract combinatorial block and then notice that the same change of variables decomposes all the translations. Thus this change of variables is analytic for all ξ outside some real semialgebraic hypersurface.

Proposition 6.19. *In the Fitting decomposition of $F^{0,1}$ under $ad(N)$ each eigenvalue appears with a uniformly bounded multiplicity $\leq \kappa$.*

Proof. The eigenvalues of $i ad(N)$ are $\mu = \pm((\omega(\xi), \nu) + \tilde{\Omega}_k)$, hence two eigenvalues may coincide only if they coincide at $\xi = 0$. Supposing that this is true we study the remaining linearly homogeneous terms. The linear terms in an eigenvalue are a translation of the finite list $\lambda^{(i)}(\xi)$, and

$$(\omega(\xi) - \omega(0), \nu \pm \nu') = \lambda^{(i)} \pm \lambda^{(j)}$$

may hold only if $\nu \pm \nu'$ is uniformly bounded. □

Notice that with each eigenvalue μ there is also the eigenvalue $-\mu$ with the same multiplicity. Let us denote by $F_\mu^{0,1}$ the generalized eigenspace relative to μ . It is easy to see that, from the first Melnikov condition, we have

Lemma 6.20.

$$\{F_\mu^{0,1}, F_{\mu'}^{0,1}\} = 0 \iff \mu + \mu' \neq 0$$

moreover $F_\mu^{0,1}$ and $F_{-\mu}^{0,1}$ are in duality under Poisson bracket.

Proof. We start from the canonical decomposition into pairs of conjugate combinatorial blocks. Two different pairs are orthogonal (under Poisson bracket) each block is isotropic and in duality with the conjugate block. If we decompose a given block (identified by the frequency ν) in generalized eigenspaces under $ad(N)$ we have, a list of non zero eigenvalues $\mu_\nu^{(i)}$ and in the conjugate space the eigenvalues $-\mu_\nu^{(i)}$. The duality pairing of Poisson bracket puts in duality the eigenspace relative to $\mu_\nu^{(i)}$ and in the conjugate space the one relative to the eigenvalue $-\mu_\nu^{(i)}$. It follows that the total generalized eigenspace relative to a given eigenvalue (which is necessarily non-zero) appearing in the sum of two conjugate blocks is isotropic and in duality with the generalized eigenspace relative to the opposite eigenvalue in the same sum of two conjugate blocks. \square

We decompose $F^{0,2}$ according to the pairs (μ, μ') and denote by $F_{\mu, \mu'}^{0,2} = F_\mu^{0,1} \cdot F_{\mu'}^{0,1}$ (the product of functions). By Leibniz rule $ad(N)$ on this space has generalized eigenvalue $\mu + \mu'$. As in §3.6, for the first step we consider the Cantor set \mathcal{C} :

$$(54) \quad |\mu_\nu^{(i)}(\xi) + \mu_{\nu'}^{(j)}(\xi)| \geq \frac{\gamma(A + r^\alpha)}{1 + |\nu + \nu'|^{\tau_0}}$$

for all $\mu_\nu^{(i)}(\xi) + \mu_{\nu'}^{(j)}(\xi) \neq 0$ where $A = 1$ if the integer $\mu_\nu^{(i)}(0) + \mu_{\nu'}^{(j)}(0) \neq 0$ and $A = 0$ otherwise. The fact that this Cantor set has positive measure follows by the same reasoning as in Corollary 2.

Proposition 6.21. *For all $\xi \in \mathcal{C}$ the operator $ad(N)$ is invertible on each $F_{\mu_\nu^{(i)}, \mu_{\nu'}^{(j)}}^{0,2}$ such that $\mu_\nu^{(i)} + \mu_{\nu'}^{(j)} \neq 0$. For all $x \in F_{\mu_\nu^{(i)}, \mu_{\nu'}^{(j)}}^{0,2}$ one has:*

$$|ad(N)^{-1}x|_{D(s,r)} \leq C \frac{|\nu + \nu'|^{\kappa^2 \tau_0 + 1}}{A + r^\alpha} |x|_{D(s,r)},$$

with C some universal constant and A defined as in (54).

Proof. The operator $ad(N)$ on $F_{\mu_\nu^{(i)}, \mu_{\nu'}^{(j)}}^{0,2}$ has generalized eigenvalue $\mu_\nu^{(i)} + \mu_{\nu'}^{(j)}$ so that if $ad(N)$ is semi-simple the result is obvious with $\kappa = 1$. In general we notice that $F_{\mu_\nu^{(i)}, \mu_{\nu'}^{(j)}}^{0,2}$ is a finite dimensional space with dimension $\leq \kappa^2$. The entries of $ad(N)$ on $F_{\mu_\nu^{(i)}, \mu_{\nu'}^{(j)}}^{0,2}$ are all bounded by $C|\nu + \nu'|$. Then, if $A = 1$ the result follows by Cramer's rule. If $A = 0$ $ad(N)$ is homogeneous of degree one in ξ , hence the entries of $ad(N)r^{-\alpha}$ are bounded by $C|\nu + \nu'|$ (recall that $\xi \leq r^\alpha$), again we apply Cramer's rule. \square

Let $[R]$ be the projection of R on the spaces $F_{\mu, -\mu}^{0,2}$.

Corollary 6.22. *The operator $ad([R])$ on $F^{(0,1)}$ is block diagonal relative to the Fitting decomposition of $ad(N)$.*

Proof. The result follows by the duality of $F_\mu^{0,1}$ and $F_{-\mu}^{0,1}$ by Poisson bracket. \square

The purpose of a KAM algorithm will now be to construct a change of variables $\Phi = e^{ad(F)} : D(s/4, r/4) \rightarrow D(s, r)$ such that $e^{ad(F)} \circ H = N^\infty + P^\infty$, where

$$N^\infty = (w^\infty, y^\infty) - \frac{1}{2} w^t M(x^\infty) J w, \quad P^\infty = \sum_{i,j: 2i+j>2} P_{ij}^\infty(x^\infty)(y^\infty)^i (w^\infty)^j$$

and M is represented on $F^{0,1}$ as a block-diagonal matrix on the blocks $F_\mu^{0,1}$ defined by the Fitting decomposition of $ad(N)$.

From a purely formal point of view at each step of the KAM algorithm, we have the Hamiltonian

$$H_k = N_k + R_k + P_k$$

where $ad(N_k)$ is block diagonal with respect to the Fitting decomposition of $ad(N)$, R_k belongs to $F^{\leq 2}$ and $P_k \in F^{>2}$. Naturally one should prove appropriate estimates to show that the sequence R_k tends to zero super-exponentially while P_k is bounded. To define H_{k+1} we solve the homological equation

$$(55) \quad ad(N_k)F_k = R_k - [R_k],$$

where $[\cdot]$ is defined with respect to the Fitting decomposition of $ad(N)$. We define $N_{k+1} := N_k + [R_k]$ which remains block diagonal with respect to the initial Fitting decomposition, although it is possible that a given eigenspace may split into different eigenspaces. The main point is that, due to the fact that the R_h are very small, the Fitting decomposition of N_k may only be a refinement of the decomposition of $ad(N)$. Indeed two different eigenspaces of $ad(N)$ are different for all the $ad(N_k)$, since the correction to the eigenvalue is small. Then formula (55) defines $F_k \in F^{\leq 2}$ uniquely for ξ on an appropriate Cantor set \mathcal{C}_k where,

$$|ad(N_k)^{-1}x|_{D(s,r)} \leq r^{-\alpha}|\nu + \nu'|^\tau |x|_{D(s,r)},$$

for all $x \in F_{\mu_{\nu}^{(i)}, \mu_{\nu'}^{(j)}}^{0,2}$.

By definition $H_{k+1} := \exp(ad(F_k))H_k$, we define $R_{k+1} := \Pi_{F^{\leq 2}} \exp(ad(F_k))H_k - N_{k+1}$ and P_{k+1} consequently.

6.23. The issue of linear stability. As we have seen in the previous paragraph, in dimension 2 and 3 the three Melnikov conditions enable us to solve the Homological equation and hence put up a KAM iteration (as discussed in §3.6). For the linear stability of the normal form we must verify whether the eigenvalues $\tilde{\Omega}$ may be all real. Indeed it is not true that for all small parameters the eigenvalues are real (as one can see even in the case of a single red edge where the condition is that the discriminant $(\xi_1 + \xi_2)^2 - 16\xi_1\xi_2 > 0$). Nevertheless the condition on the parameters for the roots to be real is given by a system of inequalities (still given by Sylvester's theory).

The question is to verify that this open region intersects our domain A_α . Let us recall briefly the Theory, (cf. [15] for a modern exposition). Given a monic polynomial $f(t) = \prod_{i=1}^n (t - x_i)$ its coefficients are up to sign the elementary symmetric functions σ_h in the x_i . Consider the *Newton functions* $\psi_h := \sum_{i=1}^n x_i^h$. There are simple recursive formulas expressing the ψ_h as polynomials in the σ_k , $k \leq h$ with integer coefficients. Consider next the *Bezoutian matrix*, that is the symmetric $n \times n$ matrix B with entries ψ_{i+j-2} . Its determinant is the *discriminant* and equals $\pm \prod_{i \neq j} (x_i - x_j)$. If the polynomial has real coefficients then b is a real symmetric matrix and its signature is the number of real roots of $f(t)$.

In particular B is positive definite if and only if all the roots are real and distinct. The condition on a symmetric matrix to be positive definite is given by the positivity of the determinants of all the principal minors. In our setting thus, for every block containing red edges we deduce a finite number of inequalities in the parameters ξ_i . The region where all these inequalities are satisfied is thus the region where all the eigenvalues are real and distinct. This region is a cone and the issue is to show that it is non-empty. This requires again a very complicated case analysis which we do not perform. We refer to §13 for a partial discussion.

In dimension $n = 2$ however we may follow the approach by [10] to further simplify the allowable blocks A and hence the matrix M' of Theorem 1.

Constraint 4 ([10]). We choose the tangential sites S so that there are no geometric blocks in Theorem 5 with more than one edge.

The meaning of this constraint is as follows: a vector $k \in \mathbb{Z}^n$ is the root of a geometric block with k edges if k satisfies a system of k linear and quadratic equations (see Formula (77), notice that the equations depend only on the abstract marked graph). The constrain above can be verified by requiring that all the 2×2 systems which identify blocks with two edges do not have any *integer* solution. Naturally such a strong constrain may hold only in dimension $n = 2$.

Proof of Corollary 1. We impose Constrain 4. For all S such that Constrain 4 holds, the matrices C_A are 2×2 and quite simple (the possible $tI - C_A$ are written in Formula (11.2)). Then all the eigenvalues can be computed explicitly. By inspection one sees that all the $\tilde{\Omega}_k$ are real in the domain:

$$D := \{ \xi \in \mathbb{R}^m \mid \forall i, j = 1, \dots, m, i \neq j, \quad \xi_i^2 + \xi_j^2 - 14\xi_i\xi_j > 0 \}.$$

Clearly D is a cone and the intersection with A_α is non empty. \square

Part 2. The algebraic combinatorial Theorems

7. A GRAPH PROBLEM

7.1. A universal graph. We need to develop some combinatorics which is useful to study the graph Λ_S introduced in §5.11.

Let us thus choose m variables e_1, \dots, e_m . Denote by

$$\Lambda := \{ \sum_{i=1}^m a_i e_i \}, \quad S^2[\Lambda] := \{ \sum_{i,j=1}^m a_{i,j} e_i e_j \}, \quad a_i, a_{i,j} \in \mathbb{Z}$$

the lattices generated by these variables e_i and that generated by the products $e_i e_j$. Set $\eta : \Lambda \rightarrow \mathbb{Z}$, $\eta(e_i) := 1$ the *augmentation*.

We define a structure of *colored marked graph* on

$$\Lambda \times \mathbb{Z}/(2) := \Lambda^+ \cup \Lambda^-; \quad \mathbb{Z}/(2) = \pm 1$$

as follows.

We take two elements $(a, \sigma), a = \sum_i m_i e_i, \quad (b, \tau), b = \sum_i n_i e_i$:

i) We join $(a, \sigma), (b, \tau)$ with an oriented black edge, marked (i, j) if

$$(56) \quad \sigma = \tau, \quad b = a + e_i - e_j, \iff a = b + e_j - e_i.$$

ii) We join $(a, \sigma), (b, \tau)$ with an unoriented red edge, marked (i, j) if

$$(57) \quad \sigma = -\tau, \quad b + a + e_j + e_i = 0.$$

Remark 7.2. In case i) we have $\eta(a) = \eta(b)$ in case ii) we have $\eta(a) + \eta(b) = -2$.

We will draw a black edge with its orientation either as horizontal or as vertical edge as

$$(58) \quad \begin{array}{ccccc} b = a + e_i - e_j & a \xrightarrow{(i,j)} b & \begin{array}{c} a \\ \downarrow (i,j) \\ b \end{array} & c \xrightarrow{(i,j)} d \\ c + d + e_j + e_i = 0 & & & & \end{array}$$

Recall that

Definition 7.3. i) A *path* p of length $\ell(p) = k$, from a vertex a to a vertex b in a graph is a sequence of vertices $p = \{a = a_0, a_1, \dots, a_k = b\}$ such that a_{i-1}, a_i form an edge for all $i = 1, \dots, k$.

The vertex a is called the *source* and b the *target* of the path.

ii) A *circuit* is a path from a vertex a to itself.

iii) A graph without circuits is called a *tree*.

It is sometimes useful to project the graph on the factor Λ and just follow the paths here.

Let $\sigma(p)$ be 1 if p has an even number of red edges and -1 if odd. We see that

Lemma 7.4. $\eta(a) = \eta(b)$ if $\sigma(p) = 1$, $\eta(a) + \eta(b) = -2$ if $\sigma(p) = -1$.

Given an integer α set $\Lambda_\alpha := \{a \in \Lambda \mid \eta(a) = \alpha\}$.

Proposition 7.5. i) Each Λ_α^+ or Λ_α^- is a connected component of the graph in which we only use black edges.

ii) For each pair of integers α, β with $\alpha + \beta = -2$ we have that the set $\Lambda_\alpha^+ \cup \Lambda_\beta^-$ is a connected component of the graph.

Proof. By the previous identity each connected component of the graph is inside one of these sets. It is clearly enough to prove i) which we easily see by induction. \square

The natural choice coming from conservation of mass is to take $\Lambda_{-1}^+ \cup \Lambda_{-1}^-$.

There are symmetries in the graph. First the symmetric group S_m of the $m!$ permutations of the elements e_i preserves the graph. Next the *sign change map* and, given any element $c \in \Lambda$, the *translation map*:

$$(59) \quad (a, \sigma) \mapsto (a, -\sigma), \quad \tau_c : (b, \sigma) \mapsto (b + \sigma c, \sigma).$$

If we want to restrict to $\Lambda_{-1} \times \mathbb{Z}/(2)$ we need to have $\eta(c) = 0$.

If A is any subgraph we set \bar{A} to be the transform of A under sign change.

In particular we may shift and possibly change sign to the graph so that any element (a, σ) is sent to $(0, +)$.

The element $(0, +)$ is called *the root*.

Definition 7.6. A *complete marked graph*, on a set $V \subset \Lambda \times \mathbb{Z}/(2)$ is the full sub-graph generated by the vertices in V .

We shall see that these are the combinatorial objects appearing in our context. By the previous remarks given such a V and an element $(a, \sigma) \in V$ we can apply translations and sign change so that the root $(0, +)$ is in the transformed graph. Thus we start by studying the connected graphs containing the root.

It is very convenient to understand this picture as follows.

Remark 7.7. A connected graph containing $(0, +)$ is completely determined by its projection in Λ . This lies in the set

$$(60) \quad \tilde{\Lambda} := \{a \in \Lambda \mid \eta(a) = \{0, -2\}\}.$$

Given a vector $a \in \tilde{\Lambda}$ we have either $\eta(a) = 0$ and then we assign to it the sign $+$ otherwise $-$. We shall then consider $\tilde{\Lambda}$ as a *universal graph*.

In particular the graphs that will appear are given by

Definition 7.8. A *non degenerate graph*, is a complete graph in $\tilde{\Lambda}$ containing 0 and other linearly independent vectors.

Remark 7.9. If we choose k linearly independent elements in $\tilde{\Lambda}$ in general the complete graph generated by 0 and these elements is not connected. The possible connected ones are easily seen to be a finite number.

We shall need in fact to consider equivalent two non degenerate graphs if we can translate one into the other. Inside $\tilde{\Lambda}$ the translations by elements c with $\eta(c) = 0$ preserve the sign, the translations by elements c with $\eta(c) = -2$ exchange the sign. They are used to *change the root*.

7.10. Geometric realization. Given a list S of m vectors $v_i \in \mathbb{C}^n$, we define a linear map

$$\pi_S = \pi : \Lambda \rightarrow \mathbb{C}^m, \quad \pi(e_i) := v_i.$$

We consider \mathbb{C}^m with the standard scalar product (NOT the Hilbert product). Set $(u)^2 := (u, u)$, $u \in \mathbb{C}^m$. Extend the map π to a linear map of the polynomials $S^2(\Lambda)$ of degree 2 in the e_i setting (cf. (40)):

$$\pi(e_i) := v_i, \quad \pi(e_i e_j) := (v_i, v_j),$$

We have $\pi(AB) = (\pi(A), \pi(B))$, $\forall A, B \in \Lambda$. We next define the linear map:

$$L^{(2)} : \Lambda \rightarrow S^2(\Lambda), \quad e_i \mapsto e_i^2.$$

Remark 7.11. Notice that we have $L^{(2)}(a) = a^2$ if and only if a equals 0 or one of the variables e_i .

Definition 7.12. Given an edge $(a, \sigma), (b, \tau)$ set $h := -\pi(b), k := -\pi(a)$. We say that the edge is *compatible* with S or π if

$$(61) \quad \begin{cases} (h)^2 - (k)^2 = (v_j)^2 - (v_i)^2 & \text{if } \sigma = \tau, \quad b - a = e_i - e_j \\ (h)^2 + (k)^2 = (v_j)^2 + (v_i)^2 & \text{if } \sigma = -\tau, \quad b + a + e_j + e_i = 0 \end{cases}.$$

Proposition 7.13. The sub-graph of $\Lambda \times \mathbb{Z}/(2)$ in which we keep only the edges compatible with S , coincides with the graph Λ_S defined in §5.

Proof. If the edge is black $b = a + e_i - e_j$ we have $h = k + v_j - v_i$ for a red edge instead we have $h + k = v_i + v_j$. \square

Remark 7.14. $\Lambda_S = \bar{\Lambda}_S$ is closed under sign change.

It is convenient to express in a unique form the previous identities as

$$(62) \quad \begin{aligned} \sigma(k)^2 - \tau(h)^2 &= \pi[L^{(2)}(\tau b - \sigma a)] \\ \iff \sigma[(k)^2 + \pi[L^{(2)}(a)]] &= \tau[(h)^2 + \pi[L^{(2)}(b)]]. \end{aligned}$$

Theorem 8. If $(a, \sigma), (b, \tau)$ are in the same connected component of Λ_S we have

$$(63) \quad \sigma\pi[a^2 + L^{(2)}(a)] = \tau\pi[b^2 + L^{(2)}(b)].$$

Corollary 7.15. A component of Λ_S is a complete subgraph (cf. 7.6) of the universal graph.

Proof. Fix an element (a, σ) of which we want to find the component. Consider the set of all elements (b, τ) such that (63) holds. They determine a complete subgraph and the component passing through (a, σ) of this graph is the required one. \square

The construction we just made is such that:

Proposition 7.16. *The map $(a, \sigma) \mapsto -\pi(a)$ maps to the graph Γ_S and it is compatible with the structure of the two graphs.*

One of the first constraints we shall impose on the v_i will make this map a graph isomorphism on each component, although infinitely many components map to the same geometric component. Given the geometric component A associated to an element $r = -\pi(\mu)$ and a component \mathcal{A} in Λ_S mapping to A , all the other components mapping to A are obtained as translates of \mathcal{A} under the subgroup of \mathbb{Z}^m kernel of π and their conjugates.

7.17. The colored marked graphs. One can also start in an abstract way returning to and expanding Definition 5.14.

Definition 7.18. A *colored marked graph* or \mathcal{M} -*graph* for short, is

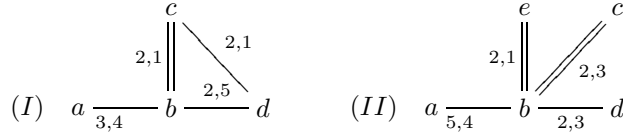
- A graph Γ (without repeated edges).
- A color *red* or *black* on each edge, displayed

$$a \xrightarrow{\text{black}} b \qquad c \xrightarrow{\text{red}} d$$

- A marking (i, j) , $1 \leq i \leq m, 1 \leq j \leq m, i \neq j$ on each oriented edge with the convention that the opposite orientation corresponds to the exchanged marking (j, i) .

The red edges are assumed to be unoriented, so for them we do not distinguish between the markings (i, j) , (j, i) .

Example 7.19.



Given a path p from a to b and another q from b to c we have the obvious:

- Opposite path $p^o := \{b = a_k, a_{k-1}, \dots, a_1 = a\}$ from b to a .
- *concatenation* $q \circ p$, a path from a to c .

Observe that a circuit $p = \{a = a_0, a_1, \dots, a_k = a\}$ induces by *rotation* a circuit from a_i to itself for every i and also an *opposite circuit* p^o .

Given a path p we define

- A *sign* $\sigma(p) = \pm$, where
 - $\sigma(p) = +$ if the path has an even number of red edges, we also say that p is *even*,
 - $\sigma(p) = -$ if the path has an odd number of red edges, we also say that p is *odd*.
- A linear combination $L(p) = \sum_i a_i e_i$, $a_i \in \mathbb{Z}$ as follows:
 - If p is an oriented edge e , i.e. $\ell(p) = 1$ marked (i, j) we set

$$(64) \qquad L(p) := \begin{cases} e_j - e_i & \text{if } e \text{ is black} \\ e_j + e_i & \text{if } e \text{ is red} \end{cases}$$

ii) If $k > 1$ let $p' := \{a_0, a_1, \dots, a_{k-1}\}$ and $e = (a_{k-1}, a_k)$. We set

$$(65) \quad L(p) = L(e) + \sigma(e)L(p').$$

The following Lemma follows from an easy induction.

Lemma 7.20.

$$(66) \quad \sigma(p^o) = \sigma(p), \quad L(p^o) = -\sigma(p)L(p).$$

$$(67) \quad \sigma(q \circ p) = \sigma(q)\sigma(p), \quad L(q \circ p) = L(q) + \sigma(q)L(p).$$

Example 7.21.

$$p := a \xrightarrow{1,2} \bullet \xrightarrow{3,2} b; \quad q := b \xrightarrow{1,3} \bullet \xrightarrow{4,2} \bullet \xrightarrow{3,4} c$$

$$q \circ p = a \xrightarrow{1,2} \bullet \xrightarrow{3,2} \bullet \xrightarrow{1,3} \bullet \xrightarrow{4,2} \bullet \xrightarrow{3,4} c$$

$$\sigma(p) = \sigma(q) = -, \quad L(p) = e_3 + e_1; \quad L(q) = 2e_4 + e_1 - e_2; \quad L(q \circ p) = 2e_4 - e_3 - e_2.$$

7.22. Compatibility. We need to restrict to smaller and smaller classes of \mathcal{M} -graphs for our analysis thus we start setting

Definition 7.23. A connected \mathcal{M} -graph A is *compatible*, if, given any two vertices a, b and two paths p_1, p_2 from a to b we have:

$$(68) \quad \sigma(p_1) = \sigma(p_2), \quad L(p_1) = L(p_2).$$

Observe that the two conditions in (68) are equivalent to saying that, given a circuit p we have

$$\sigma(p) = +, \quad L(p) = 0, \quad \forall p \text{ a circuit.}$$

For instance the graph (I) of example 7.19 is not compatible.

Assumption 1. *All the graphs we consider are compatible*

We apply this assumption as follows. Choose a vertex r of a compatible connected graph A which we call the *root*. Given any vertex a and a path p from r to a we set

$$\sigma_r(a) := \sigma(p), \quad L_r(a) := L(p).$$

Let us take as root another vertex s , and let q be a path from s to r . We clearly have from Lemma 7.20:

$$(69) \quad \sigma_s(a) = \sigma_r(a)\sigma_r(s), \quad L_s(a) = L_r(a) - \sigma_r(a)\sigma_r(s)L_r(s)$$

Assumption 2. We restrict to those graphs for which the linear forms $L_r(a)$ are all *different*.

Observe that this assumption does not depend on the choice of the root. We have

$$(70) \quad \eta(L_r(a)) = \begin{cases} 0 & \text{if } \sigma(a) = + \\ 2 & \text{if } \sigma(a) = - \end{cases}$$

Theorem 9. *In a compatible graph A satisfying the previous assumption, the mapping $\lambda : a \mapsto -L_r(a)$ embeds A as a subgraph of the universal graph $\tilde{\Lambda} = \{a \in \Lambda \mid \eta(a) = \{0, -2\}\}$ (cf. 7.7).*

Proof. By Assumption 2 we have an embedding at the level of vertices. We need to show that this is compatible with the edges. Take an edge $e = (a, b)$ in A , we know that we have $L_r(b) = L_r(e) + \sigma(e)L_r(a)$ hence if e is red marked (i, j) we have $0 = e_i + e_j - L_r(a) - L_r(b)$. If e is black marked (i, j) we have $0 = e_j - e_i + L_r(a) - L_r(b)$ or $-L_r(b) = e_i - e_j - L_r(a)$. \square

This embedding maps r to $(0, +)$. We could have defined more general embeddings but they can be obtained from this by translations and sign changes. From now on, unless it is necessary, we shall drop the symbol r in $\sigma_r(a), L_r(a)$.

Remark 7.24. Assume that, for a given root r of a graph A we have k vertices marked $+$ and h marked $-$. If we change the root to a root s , and s is marked $+$ we still have k vertices marked $+$ and h marked $-$, if s is marked $-$ we have $h - 1$ vertices marked $+$ and $k + 1$ marked $-$.

The embedding changes by the translation by $L_r(s)$ from (69) and (59)

The main reason of this paragraph is the following:

Remark 7.25. Let A be a component of the geometric graph, then to A is associated in a natural way an abstract colored marked graph. If we choose a root and perform the embedding given by Theorem 9 we see that we have lifted the geometric graph to the graph Λ_S inverting the map π .

We shall prove that, under the generic assumption, this lift is an isomorphism to a connected component of Λ_S over A . By abuse of notations we shall denote often by A also this component. All other components over A are obtained from this by translations with elements in $\text{Ker}(\pi)$.

7.26. Realizing the graphs.

Definition 7.27. A realization of an \mathcal{M} -graph is obtained by composition of the embedding, a translation and a realization as in the previous section $a \mapsto -\pi(\sigma(a)r - L_r(a)) = -\sigma(a)\pi(r) + \pi(L_r(a))$.

A realization is called *integral* if all the vectors v_i have integer coordinates and all the x_a lie in the lattice $\pi(\mathbb{Z}^m) = \mathcal{L}_v$ generated by the v_i .

Remark 7.28. We have defined a realization by the choice of a root and a translation. Taking a different root we may obtain the same realization by changing appropriately the translation.

We suppose that our graphs are rooted and drop the subscript r .

Definition 7.29. Given a graph A we then set for a vertex a

$$(71) \quad \begin{aligned} V(r) &:= 0, N(r) := 0, V(a) := \pi(L(a)), \\ L^{(2)}(a) &:= L^{(2)}(L(a)), N(a) := \pi(L^{(2)}(a)). \end{aligned}$$

Then a realization of the graph, given the vectors v_i consists in assigning to each vertex a a vector $x_a \in \mathbb{C}^s$ so that, setting $x := -\pi(r)$, the following constraints are verified:

$$(72) \quad x_a = \sigma(a)x + V(a), \quad (x_a)^2 = \sigma(a)(x)^2 + N(a).$$

Observe that, if the v_i are integral, the realization is integral if and only if $x \in \mathcal{L}_v$.

The meaning of a realization is really a reformulation of the Properties used in Lemma 4.25.

In fact suppose that we have a path p from r to a and an edge e from a to b . We have by definition (65)

$$L(b) = \begin{cases} e_j - e_i + L(a), & \text{if } e \text{ is black marked } e_j - e_i \\ e_j + e_i - L(a), & \text{if } e \text{ is red marked } e_j + e_i \end{cases}$$

$$(73) \quad V(b) = \pi(L(b)) = \begin{cases} v_j - v_i + V(a), & \text{if } e \text{ is black marked } e_j - e_i \\ v_j + v_i - V(a), & \text{if } e \text{ is red marked } e_j + e_i \end{cases}$$

$$(74) \quad x_a = \sigma(a)x + V(a), x_b = \sigma(b)x + V(b),$$

thus

$$x_b = \sigma(b)x + v_j - v_i + V(a) = \sigma(a)x + v_j - v_i + V(a) = v_j - v_i + x_a$$

if e is black marked $e_j - e_i$

$$x_b = \sigma(b)x + v_j + v_i - V(a) = -\sigma(a)x + v_j + v_i - V(a) = v_j + v_i - x_a$$

if e is red marked $e_j + e_i$. For the scalar products

$$(75) \quad (x_a)^2 = \sigma(a)(x)^2 + N(a), (x_b)^2 = \sigma(b)(x)^2 + N(b),$$

$$(76) \quad N(b) = \begin{cases} (v_j)^2 - (v_i)^2 + N(a), & \text{if } e \text{ is black marked } e_j - e_i \\ (v_j)^2 + (v_i)^2 - N(a), & \text{if } e \text{ is red marked } e_j + e_i \end{cases}$$

thus finally we have

Theorem 10. *The constraints (72) are equivalent to the recursive identities, (which correspond to simple steps in the graph Γ_S):*

$$(77) \quad \begin{cases} x_b = v_j - v_i + x_a, & (v_j)^2 - (v_i)^2 = (x_b)^2 - (x_a)^2, \\ \text{if } e \text{ is black marked } e_j - e_i \\ x_b = v_j + v_i - x_a, & (v_j)^2 + (v_i)^2 = (x_a)^2 + (x_b)^2, \\ \text{if } e \text{ is red marked } e_j + e_i \end{cases}$$

In particular the definition of realization does not depend on the choice of the root r .

It is convenient to reformulate the equations (72) as:

$$(78) \quad \begin{aligned} \sigma(a)(x)^2 + N(a) &= (\sigma(a)x + V(a), \sigma(a)x + V(a)) \\ &= (x)^2 + 2\sigma(a)(x, V(a)) + (V(a))^2. \end{aligned}$$

Let $L(a) = \sum_h a_h e_h$ so that $V(a) = \sum_h a_h v_h$.

We have: $(V(a))^2 - N(a) = \sum_h (a_h^2 - a_h)(v_h)^2 + 2 \sum_{h < k} a_h a_k (v_h, v_k)$. Thus denote by

$$(79) \quad C(a) := \sum_h \binom{a_h}{2} (v_h)^2 + \sum_{h < k} a_h a_k (v_h, v_k) = \frac{1}{2}((V(a))^2 - N(a)).$$

Formula (78) becomes, using (79)

Theorem 11.

$$(80) \quad -C(a) = \begin{cases} (x, V(a)) & \text{if } \sigma(a) = + \\ -(x, V(a)) + (x)^2 & \text{if } \sigma(a) = - \end{cases}$$

We need one further definition

Definition 7.30. A graph A is *allowed* if for generic choices of integral vectors v_i it admits an integral realization.

The word *generic* is used as in algebraic geometry. It means that the property holds outside the zeros of some polynomial equation in the coordinates of the v_i .

One of our main tasks is to study such graphs.

7.31. Relations. Let us introduce a

Definition 7.32.

- A graph A with $k + 1$ vertices is said to be of *dimension* k .
- The lattice \mathbb{Z}_r^m generated by the elements $L(a)$ as a runs over the vertices for a given choice of a root r is independent of the root. We call its dimension the *rank*, $\text{rk}A$, of the graph A .
- If the rank of A is strictly less than the dimension of A we say that A is *degenerate*.

Proof of item 2. If we change the root from r to another a we have by Formulas (7.20) that.

$$L_s(a) = L_r(a) - \sigma(s)\sigma(a)L_r(s).$$

This shows that $\mathbb{Z}_s^m \subset \mathbb{Z}_r^m$ and of course also the converse is true by exchanging the two roles.

□

If A is degenerate then there are non trivial relations $\sum_a n_a \sigma(a) L(a) = 0$, $n_a \in \mathbb{Z}$ among the elements $L(a)$.

It is also useful to choose a maximal tree T in A . There is a triangular change of coordinates from the $L(a)$ to the markings of T . Hence the relation can be also expressed as a relation between these markings.

If we are given a realization $\pi : e_i \rightarrow v_i$ of the graph we must have, for every relation $\sum_a n_a \sigma(a) L(a) = 0$, $n_a \in \mathbb{Z}$ that $\sum_a n_a \sigma(a) V(a) = 0$ and, using Formula (70)

$$(81) \quad 0 = \sum_{a, \mid \sigma(a)=-} n_a.$$

Applying Formula (80) we deduce that we must have

$$(82) \quad \sum_a n_a C(a) = -(x, \sum_a n_a \sigma(a) V(a)) - \left[\sum_{a, \mid \sigma(a)=-} n_a \right] (x)^2$$

hence

$$(83) \quad \sum_a n_a C(a) = -(x, \sum_a n_a \sigma(a) V(a)) = 0.$$

Let us thus set

$$(84) \quad G(a) := \sum_h \binom{a_h}{2} e_h^2 + \sum_{h < k} a_h a_k e_h e_k = \frac{1}{2} (L(a)^2 - L^{(2)}(a)).$$

We have $C(a) = \pi(G(a))$. Remark that also $\sum_a n_a \sigma(a) L^{(2)}(a) = 0$.

We have

$$\begin{aligned} \sum_a n_a G(a) &= \frac{1}{2} \sum_a n_a L(a)^2 - \frac{1}{2} \sum_a n_a L^{(2)}(a) \\ \sum_a n_a L^{(2)}(a) &= 2 \sum_{a, \sigma(a)=-} n_a L^{(2)}(a). \end{aligned}$$

and

$$0 = \sum_a n_a C(a) = \pi \left(\sum_a n_a G(a) \right) = \frac{1}{2} \pi \left(\sum_a n_a L(a)^2 - \sum_{a \mid \sigma(a)=-} n_a L^{(2)}(a) \right).$$

Definition 7.33. If $\sum_a n_a L(a)^2 - \sum_{a \mid \sigma(a)=-} L^{(2)}(a) \neq 0$ then the equation (82) is a non trivial constraint, and we say that the graph has an *avoidable resonance*.

Remark 7.34. If we have an avoidable resonance then for a generic choice of the $S := \{v_i\}$ the graph is not realized by π_S .

We arrive now at the main Theorem of the section:

Theorem 12. *Given a compatible graph of rank k then either it has $k + 1$ vertices or it produces an avoidable resonance.*

Proof. Let $q + 1$ be the number of vertices. By compatibility the rank of the graph equals the rank of a maximal tree which has q edges, hence $q \geq k$.

Assume by contradiction that $q > k$. Choose a root, we can choose $k + 1$ vertices (a_0, a_1, \dots, a_k) so that we have a non trivial relation $\sum_a n_a \sigma(a) L(a_i) = 0$ and the elements $L(a_i)$, $i = 1, \dots, k$ are linearly independent.

We claim that $\sum_a n_a L(a)^2 - \sum_{\sigma(a)=-} n_a L^{(2)}(a) \neq 0$ and thus we have produced an avoidable resonance. Suppose by contradiction that $\sum_a n_a L(a)^2 - \sum_{\sigma(a)=-} n_a L^{(2)}(a) = 0$. Assume first that all these vertices are marked $+$, we have then $\sum_a n_a L(a)^2 = 0$. Similarly, if they are all marked $-$ we have $-\sum_a n_a L(a) = \sum_a n_a \sigma(a) L(a) = 0$ and also $\sum_a n_a L^{(2)}(a) = 0$ so again $\sum_a n_a L(a)^2 = 0$.

We can consider thus the elements $x_i := L(a_i)$, $i = 1, \dots, k$ as new variables and then we write the relation as

$$0 = L(a_{k+1}) + \sum_{i=1}^k p_i x_i, \implies \left(\sum_{i=1}^k p_i x_i \right)^2 + \sum_{i=1}^k p_i x_i^2 = 0.$$

Now $\sum_{i=1}^k p_i x_i^2$ does not contain any mixed terms $x_h x_k$, $h \neq k$ therefore this equation can be verified if and only if the sum $\sum_{i=1}^k p_i x_i$ is reduced to a single term $p_i x_i$, and then we have $p_i = -1$ and $L(a_0) = L(a_i)$, contrary to the second assumption.

Finally assume we have in the relation m vertices marked $+$ and n marked $-$. We think of the elements $L(a_i)$ as linear functions in some variables y_i . Set $\underline{y} = \underline{e} + \underline{y}'$, where $\underline{e} := (1, 1, \dots, 1)$. Assume

$$\begin{aligned} \sum_{i=1}^m a_i u_i(\underline{y}) - \sum_{j=1}^n b_j v_j(\underline{y}) &= 0 \\ u_i(\underline{e}) &= 0, \quad v_j(\underline{e}) = 2 \implies \sum_j b_j = 0. \end{aligned}$$

Now assume

$$\sum_{i=1}^m a_i u_i^2 - \sum_{j=1}^n b_j [v_j^{(2)} - v_j^2] = 0$$

For any linear form L ,

$$L(\underline{y}) = L(\underline{y}') + L(\underline{e}), \quad L^{(2)}(\underline{y}) = L(\underline{e}) + 2L(\underline{y}') + L^{(2)}(\underline{y}'),$$

in particular

$$u_i(\underline{y}') = u_i(\underline{y}), \quad v_j(\underline{y}') = v_j(\underline{y}) - 2.$$

$$v_j^{(2)}(\underline{y}) - v_j(\underline{y})^2 = 2 + 2v_j(\underline{y}') + v_j^{(2)}(\underline{y}') - (2 + v_j(\underline{y}'))^2 = -2 - 2v_j(\underline{y}') + v_j^{(2)}(\underline{y}') - v_j(\underline{y}')^2.$$

The relation becomes

$$\begin{aligned}
0 &= \sum_{i=1}^m a_i u_i^2(\underline{y}') - \sum_{j=1}^n b_j [-2 - 2v_j(\underline{y}') + v_j^{(2)}(\underline{y}') - v_j(\underline{y}')^2] \\
&\implies \sum_{i=1}^m a_i u_i^2(\underline{y}') - \sum_{j=1}^n b_j [v_j^{(2)}(\underline{y}') - v_j(\underline{y}')^2] \\
&= \sum_{j=1}^n b_j [-2 - 2v_j(\underline{y}')] = -2 \sum_{j=1}^n b_j v_j(\underline{y}').
\end{aligned}$$

The left hand side is homogeneous of degree 2 and the right of degree 1. This implies $\sum_{j=1}^n b_j v_j(\underline{y}') = 0$ and we are back in the previous case. \square

8. MAIN GEOMETRIC THEOREM

8.1. Determinantal varieties. In this section we think of a marking $\pm v_i \pm v_j$ or more generally of an expression $\sum_{i=1}^m a_i v_i$ as a *map* from $V^{\oplus m}$ to V . Here V is a vector space where the v_i belong. Thus a list of k markings is thought of as a map $\rho : V^{\oplus m} \rightarrow V^{\oplus k}$. Such a map is given by a $k \times m$ matrix.

When $\dim(V) = n$ we shall be interested in particular in n -tuples of markings. In this case we have

Lemma 8.2. *An n -tuples of markings $m_i := \sum_j a_{ij} v_j$ is formally linearly independent – that is the $n \times m$ matrix of the a_{ij} has rank n – if and only if the associated map $\rho : V^{\oplus m} \rightarrow V^{\oplus n}$ is surjective.*

We may identify $V^{\oplus n}$ with $n \times n$ matrices and we have the determinantal variety D_n of $V^{\oplus n}$ defined by the vanishing of the determinant and formed by all the n -tuples of vectors v_1, \dots, v_n which are linearly dependent. The variety D_n defines a similar irreducible variety $D_\rho := \rho^{-1}(D_n)$ in $V^{\oplus m}$ which depends on the map ρ . We need to see when different lists of markings give rise to different determinantal varieties in $V^{\oplus m}$.

Lemma 8.3. *Given $\rho : V^{\oplus m} \rightarrow V^{\oplus n}$, a vector $a \in V^{\oplus m}$ is such that $a + v \in D_\rho$, $\forall v \in D_\rho$ if and only if $\rho(a) = 0$.*

Proof. Clearly if $\rho(a) = 0$ then a satisfies the condition. Conversely if $\rho(a) \neq 0$, we think of $\rho(a)$ as a non zero matrix B and it is easily seen that there is a matrix $X \in D_n$ such that $B + X \notin D_n$. \square

Let $\rho_1, \rho_2 : V^{\oplus m} \rightarrow V^{\oplus n}$ be two surjective maps given by two $n \times m$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$; $a_{i,j}, b_{i,j} \in \mathbb{C}$.

Theorem 13. $\rho_1^{-1}(D_n) = \rho_2^{-1}(D_n)$ if and only if the two matrices A, B have the same kernel.

Proof. First observe that the two matrices A, B have the same kernel if and only if ρ_1, ρ_2 have the same kernel.

By Lemma 8.3, if $\rho_1^{-1}(D_n) = \rho_2^{-1}(D_n)$ then the two matrices A, B have the same kernel. Conversely if the two matrices A, B have the same kernel we can write $B = CA$ with C invertible. Clearly $CD_n = D_n$ and the claim follows. \square

We shall also need the following well known fact:

Lemma 8.4. *Consider the determinantal variety D given by $d(X) = 0$ of $n \times n$ complex matrices of determinant zero. The real points of D are Zariski dense in D .³*

Proof. Consider in D the set of matrices of rank exactly $n - 1$. This set is dense in D and obtained from a fixed matrix (for instance the diagonal matrix I_{n-1} with all 1 except one 0) by multiplying $AI_{n-1}B$ with A, B invertible matrices. If a polynomial f vanishes on the real points of D then $F(A, B) := f(AI_{n-1}B)$ vanishes for all A, B invertible matrices and real. This set is the set of points in \mathbb{R}^{2n^2} where a polynomial (the product of the two determinants) is non zero. But a polynomial which vanishes in all the points of \mathbb{R}^m where another polynomial is non zero is necessarily the zero polynomial. So f vanishes also on complex points. This is the meaning of Zariski dense. \square

8.5. Special graphs. Let $V := \mathbb{C}^n$ so $V^{\oplus m} = \mathbb{C}^{mn}$. Take a connected \mathcal{M} -graph with $n + 2$ vertices, assume that for generic v_i this graph is realizable. By Theorem 12 this implies that the rank of this graph is $n + 1$. Choose in this graph a root, then the variety R_A of realizations is given by the solutions of equations (80), which we think as equations in both the variables of the vector x (corresponding to the root) and also of the parameters v_i .

The variety R_A maps to the space \mathbb{C}^{mn} of m -tuples of vectors v_i , call $\theta : R_A \rightarrow \mathbb{C}^{mn}$ the projection map. For a given choice of the v_i the fiber of this map θ is the set of realizations.

Proposition 8.6. *Under the previous hypotheses there is an irreducible hypersurface W of \mathbb{C}^{mn} such that the map θ is invertible on $\mathbb{C}^{mn} \setminus W$ with inverse a polynomial map.*

Assume for a moment the validity of Proposition 8.6.

Theorem 14. *Consider a graph A which contains at least $n + 1$ edges such that the markings are linearly independent, and assume A is allowable.*

Then for generic v_i 's it has a unique realization in the special component.

Proof. Consider the system on $n + 1$ linear and quadratic equations in the variables x, v_i defining the variety R_A . We are assuming that we have a solution $x = F(v)$ which is a polynomial in v_1, \dots, v_m . A degree consideration shows that F is homogeneous and linear in these variables. In fact we have since the right hand side of the equations 80 are quadratic, we have $F(\lambda v) = \lambda F(v)$.

Now the equations 80 are invariant under the action of orthogonal matrices, i.e. if A is orthogonal $F(Av_1, \dots, Av_m) = AF(v_1, \dots, v_m)$. Since the space V of the v_i (which we may take as complex) is irreducible under the orthogonal group, a linear map $V \rightarrow V$ commuting with the orthogonal group is a scalar so it follows that any linear map $V^{\oplus m} \rightarrow V$ commuting with the orthogonal group is of the form $F(v_1, \dots, v_m) = \sum_{a=1}^m c_a v_a$ for some constants c_a .

Now $x = \sum_{a=1}^m c_a v_a$ is the point of the realization corresponding to the root and so it satisfies either $(x, v_j - v_i) = (v_i, v_j - v_i)$ for some i, j so $x = v_i$ or the quadratic equation $(x - v_i, x - v_j) = 0$ from which $x = v_i$ or v_j .

Once we know that one point in the realization is in the special component we have proved (see 5.9) that the whole tree is special and realized in this component. \square

³this means that a polynomial vanishing on the real points of D vanishes also on the complex points.

8.7. Proof of Proposition 8.6, black edges. Let us first study the case of all black edges. The next is standard and follows immediately from the unique factorization property of polynomial algebras:

Theorem *Let W be a subvariety of \mathbb{C}^N of codimension ≥ 2 , let F be a rational function on \mathbb{C}^N which is holomorphic on $\mathbb{C}^N \setminus W$, then F is a polynomial.*

We have a list of $n+1$ -linear equations $(x, a_i) = b_i$ with the markings $a_i = \sum_j a_{ij}v_j$ formally linearly independent. The hypotheses made imply that any n of these equations are generically linearly independent. Call C the matrix with rows the vectors a_i .

Therefore on the open set where n of these are linearly independent the solution to the system is unique and given by Cramer's rule.

In order to complete our statement it is enough to show that the subvariety W where any n of these equations are linearly dependent has codimension ≥ 2 . The condition to be in W is that all the determinants of all the maximal minors should vanish.

Each one of these determinants is an irreducible polynomial so it defines an irreducible hypersurface. It is thus enough to see that these hypersurfaces are not all equal. This follows from Theorem 13, indeed by hypothesis the matrix $B = (a_{ij})$ has rank $n+1$. All the matrices obtained by B dropping one row define the various determinantal varieties, the fact that these varieties are not equal depends on the fact that the matrices cannot have all the same kernel (otherwise the rank of B is $\leq n$).

8.8. Proof of Proposition 8.6, red edges. When we also have red edges we see that the equations(80) are clearly equivalent to a system on n linear equations associated to formally linearly independent markings, plus a quadratic equation chosen arbitrarily among the ones appearing in (80). We then put as constraint the non vanishing of the determinant of the linear system we have found. Thus a realization is obtained by solving this system and, by hypothesis, such solution satisfies also the quadratic equation.

Let P be the space of functions $\sum_{i=1}^m c_i v_i$, $c_i \in \mathbb{R}$ and (P, P) their scalar products. Assume we have a list of n equations $\sum_{j=1}^m a_{ij}(x, v_j) = (x, t_i) = b_i$ with the $t_i = \sum_{j=1}^m a_{ij}v_j$ linearly independent in the space P and $b_i = \sum_{h,k} a_{h,k}^i(v_h, v_k) \in (P, P)$.

Solve these equations by Cramer's rule considering the v_i as parameters. Write $x_i = f_i/d$, where $d(v) := \det(A(v))$ is the determinant of the matrix $A(v)$ with rows t_i , $f_i(v)$ is also a determinant of another matrix $B(v)$ both depending polynomially on the v_i . We have thus expressed the coordinates x_i as rational functions of the coordinates of the v_i . The denominator is an irreducible polynomial vanishing exactly on the determinantal variety of the v_i for which the matrix of rows t_j , $j = 1, \dots, n$ is degenerate.

Lemma 8.9. *Assume there are two elements $a \in P, b \in (P, P)$ such that $(x)^2 + (x, a) + b = 0$ holds identically (in the parameters v_i); then x is a polynomial in the v_i .*

Proof. Substitute $x_i = f_i/d$ in the quadratic equation and get

$$d^{-2} \left(\sum_i f_i^2 \right) + d^{-1} \sum_i f_i a_i + b = 0, \implies \sum_i f_i^2 + d \sum_i f_i a_i + d^2 b = 0.$$

Since $d = d(v) = \det(A(v))$ is irreducible this implies that d divides $\sum_i f_i^2$.

For those $v_i \in \mathbb{R}^n$ for which $d(A(v)) = 0$, since the f_i are real we have $f_i(v) = 0, \forall i$, so f_i vanishes on all real solutions of $d(A(v)) = 0$. These solutions are Zariski dense, by Lemma 8.4, so $f_i(v)$ vanishes on all the v_i solutions of $d(A(v)) = 0$ and $d(v)$ divides $f_i(v)$, hence x is a polynomial. □

9. PROOF OF THEOREMS 4 AND 5

9.1. The resonance inequalities. We are now ready to explain which restrictions we impose on the vectors v_i in order to say that the vectors have been chosen *in a generic way*.

- 1) We impose Constraint 1.
- 2) We require that any n of the v_i are linearly independent.
- 3) We want that any linear combination of the v_i arising from an odd circuit of length $\leq n$ is non-zero.
- 4) We list all degenerate graphs with $k + 1$, $k \leq n + 1$ vertices, we list all the avoidable resonance equations and impose that the vectors v_i do not satisfy any of these equations, this is a set of quadratic inequalities.
- 5) Finally for each graph with $n + 1$ formally linearly independent markings, we impose the following relations. From equations 80 we select n linear equations and impose that the corresponding determinant is non-zero. This allows us to apply Theorem 6.

We are now ready to prove Theorems 4 and 5. The first four constraints control the components with at most n linearly independent edges.

The fifth constraint handles graphs with $n + 2$ vertices.

Let us consider a connected component A of the graph Γ_X . If we choose an element $r \in A$ and a $\mu \in \mathbb{Z}^m$ with $r = -\pi(\mu)$ we see that

Lemma 9.2. *The connected component B of the graph Λ_S containing μ maps surjectively to A .*

Proof. Clearly any edge in A can be lifted to an edge in B . □

Theorem 4 follows from the more precise

Proposition 9.3. *If A is different from the special component, then B has rank $\leq n$ and is non degenerate and $-\pi$ maps B isomorphically to A .*

Proof. Assume first that B has rank $\leq n$, we have that B has to be non-degenerate since we have imposed that the v_i do not satisfy the resonance equations. In particular B has at most $n + 1$ vertices. Then the constraints that we have required imply first that $-\pi$ is bijective on the vertices, in fact if we had two vertices in B which map to the same vertex k in A we consider a path connecting them we have that either $2k = \sum_i a_i v_i$ if the path is odd and then we can exclude this by imposing that the quadratic equation satisfied by k is not satisfied by the v_i , or we get $\sum_i a_i v_i = 0$ and again we may exclude this, in fact since B varies on graphs with at most $n + 1$ vertices all these are a finite number of inequalities. The isomorphism at the level of edges follows from Corollary 7.15.

So it only remains to show that for generic v_i we cannot have B of rank $\geq n + 1$. This follows from Theorem 14. □

Proof of Theorem 5. *i)* Let A be a connected component of the graph Γ_S , let B be a corresponding component in Λ_S so that $A = -\pi(B)$. By proposition 9.3 B has at most $n + 1$ vertices.

ii) Two points are connected by a red edge if they belong to a sphere S_{ij} with diameter v_i, v_j and there are finitely many integral points on such a sphere.

iii) Each component of Γ_S is a realization of an abstract marked graph, which encodes the equations that the root x_r should satisfy. In the case of black edges the equations are all linear and the general solution is given by adding the solutions of the associated

homogeneous system. Finally, there are only a finite number of abstract marked graphs with at most $n + 1$ vertices. \square

Remark 9.4. In this general discussion we have restricted the possible types of components that we can have in the geometric graph Γ_X . In fact it may very well be that some of the possible components which are allowed do not appear.

This depends upon the fact that our conditions are algebraic and we only claim that a certain system of equations has a solution. But, for contributing a component to the graph Γ_X , this solution must lie in the lattice spanned by the v_i . This we have not tried to discuss. In fact in dimension 2, in the paper [10], Geng, You and Xu use arithmetic conditions to exclude all graphs with 3 vertices.

9.5. $m = \infty$. It is easy to see that we can also construct infinite sequences of integral vectors v_i satisfying all constraints. For this recall the known, fact. Let $f(x_1, \dots, x_p)$ be a polynomial with integer coefficients. Assume that all coefficients of the polynomial are $< C$ in absolute value, and all exponents are $< D$, with C, D two positive integers. Consider the sequence $\mathcal{L} := a_i := C^{D^i} < i = 1, 2, \dots$. Then

Lemma 9.6. *For every choice of p distinct elements a_{i_1}, \dots, a_{i_p} in this list we have*

$$f(a_{i_1}, \dots, a_{i_p}) \neq 0.$$

Proof. Consider the monomials appearing in f .

$$x_1^{h_1} x_2^{h_2} \dots x_p^{h_p} \mapsto C^{\sum_{j=1}^p h_j D^{a_j}}.$$

Since all indices $h_i < D$ we have that different monomials of f give rise to a different exponent $\sum_{j=1}^p h_j D^{a_j}$.

Now the polynomial gives a linear combination of integers a_i with $|a_i| < C$ (the coefficients) times distinct elements $C^{D^{a_i}}$. By the uniqueness of the expression of a number in a given basis we deduce the claim. \square

We now apply this to our setting, we take C, D bigger than the coefficients of all constraints in dimension n , similar bigger than the exponents in these constraints.

If we now partition in any desired way the list \mathcal{L} into disjoint sublists each made of n elements they form an allowable infinite list.

10. THE MATRICES

In this and the following sections we discuss the combinatorial and algebraic features of the matrices appearing in the blocks of $ad(N)$ in order to complete the proof of Theorems 6, 7 and Lemma 6.17.

10.1. Combinatorial matrices. We now discuss the matrices $C_{\mathcal{A}}$ introduced in Definition 6.7. The vertices of \mathcal{A} index a basis of this block. Given a vertex (a, σ) if $a = \sum_i m_i e_i$ we set

$$a(\xi) := \sum_i m_i \xi_i.$$

Lemma 10.2. *The entries of the matrix $C_{\mathcal{A}}$, over the indexing set of the vertices of \mathcal{A} , are:*

- *In the diagonal at the vertex (a, σ) equals $-\sigma a(\xi)$.*

- At the position $(a, \sigma), (b, \tau)$ we put 0 unless they are connected by an oriented edge $e = ((a, \sigma), (b, \tau))$ marked with (i, j) . In this case we place

$$(85) \quad C(e) := 2\tau \sqrt{\xi_i \xi_j}.$$

Proof. Let $(\nu, +) \in \mathcal{A}$ correspond to the root $x(A) = -\pi(\nu)$. Take another element $a = (\mu, \sigma) \in \mathcal{A}$ such that $-\pi(\mu) = x_a \in A$ and $\sigma(a) = \sigma$. From Formula 71 and 72 and Proposition 9.3 we have $\mu = \sigma\nu - L(a)$ hence $-\sigma a(\xi) = -\sigma(\xi, \mu) = -(\xi, \nu) + \sigma(\xi, L(a))$ which is the diagonal entry of $C_{\mathcal{A}}$ by Formula (46) and Definition 6.7. \square

We need to see the behavior under the symmetric group, translation and sign change. Under the symmetric group we just permute the variables ξ_i .

Theorem 15.

$$(86) \quad C_{\tau_c(\mathcal{A})} = c(\xi)I + C_{\mathcal{A}}, \quad C_{\bar{\mathcal{A}}} = -C_{\mathcal{A}}.$$

Proof. If we translate \mathcal{A} , to $\tau_c(\mathcal{A})$ the edge $e = ((a, \sigma), (b, \tau))$ becomes $\tau_c(e) = ((a + \sigma c, \sigma), (b + \tau c, \tau))$ so

$$C(\tau_c(e)) = C(e), \quad \sigma(a + \sigma c)(\xi) = \sigma(a)(\xi) + c(\xi).$$

Similarly for sign change. \square

We denote by

$$\chi_{\mathcal{A}}(t) := \det(t - C_{\mathcal{A}})$$

the characteristic polynomial of $C_{\mathcal{A}}$.

Remark 10.3. Notice that, if e is red, in position (b, a) we have $\sigma(a)2\sqrt{\xi_i \xi_j} = -\sigma(b)2\sqrt{\xi_i \xi_j}$.

If we change the root s the matrix is changed as $C_{\mathcal{A}}^s = L(s)(\xi) + \sigma(a)C_{\mathcal{A}}^r$.

In particular $C_{\mathcal{A}}$ is symmetric if and only if there are no red edges.

These are the matrices appearing in our Hamiltonian, but we can immediately change them as follows. Choose a maximal tree in the graph and a root, then every vertex is connected by a unique minimal path. Given a vertex a and the minimal path $p = (r = a_0, a_1, \dots, a_k = a)$ from the root r to a and we set $k = \ell(a)$. We next set

$$(87) \quad D(r) := 1, \quad D(a) := \prod_{i=0}^{k-1} C(a_i, a_{i+1})$$

This defines a diagonal matrix D .

Proposition 10.4. Set $\tilde{C}_{\mathcal{A}} := DC_{\mathcal{A}}D^{-1}$:

i) If (a, b) is an edge in the tree we have

$$(88) \quad \tilde{C}_{\mathcal{A}}(a, b) = \begin{cases} \sigma(b) & \text{if } \ell(b) = \ell(a) + 1 \\ C(a, b)^2 = 4\xi_i \xi_j & \text{if } \ell(b) = \ell(a) - 1 \end{cases}$$

ii) If $e = (a, b)$ is not an edge in the tree (but it is an edge in the graph), we have that $\tilde{C}_{\mathcal{A}}(a, b)$ is a constant times a monomial in the variables $\xi_i^{\pm 1}$.

Proof. (i) This is from the definition.

(ii) This comes from the circuit that shows that modulo 2 the two elements $L(a) = L(b)$ which takes away the squares. \square

Corollary 10.5. For every allowable \mathcal{A} the characteristic polynomial of $C_{\mathcal{A}}$ has as coefficients polynomials in the variables ξ_i .

By the previous Theorem the square roots disappear.

Conjecture 1 For every allowable \mathcal{A} the characteristic polynomial $\chi_{\mathcal{A}}(t)$ of $C_{\mathcal{A}}$ is irreducible as polynomial with coefficients in $\mathbb{C}[\xi_1, \dots, \xi_m]$.

It is clear that the irreducibility property is invariant under all the symmetries, the symmetric group, translation and sign change, so the statement needs to be checked only for finitely many \mathcal{A} . In the next section we discuss dimension 3. We take as \mathcal{A} always a graph containing 0 and we assume that 0 has sign +, (cf. Definition 7.8).

10.6. The method. We have seen in Theorem 5 that the graphs we need to consider are complete subgraphs \mathcal{A} of the graph Λ_S , constructed as follows. Take a linearly independent list of k vectors ν_i where k is the rank. Consider the complete graph \mathcal{A} generated by $0, \nu_1, \dots, \nu_k$.

We need to study those \mathcal{A} which are connected. In Theorem 5 we have seen that the connected components of Λ_S are obtained from such a graph by translation and sign change.

The goal of the rest of the paper is to prove the two Conjectures which allow us to deduce the separation of eigenvalues and thus the second Melnikov equation. We have at our disposal several theoretical tools which at the moment are not sufficient to treat all cases. This is why the possible obvious inductions are not available at the moment and we need to perform a rather tedious detailed case analysis. In principle this can be checked by a finite algorithm in all dimensions but we have not tried to write the necessary code.

The main ingredient of a possible induction is:

Theorem 16. *Take an allowed graph \mathcal{A} and compute its characteristic polynomial $\chi_{\mathcal{A}}(t)$. When we set a variable $\xi_i = 0$ we obtain the product of the polynomials $\chi_{\mathcal{A}_i}(t)$ where the \mathcal{A}_i are the connected components of the graph obtained from \mathcal{A} by deleting all the edges in which i appears as index, with the induced markings (with $\xi_i = 0$).*

Proof. This is immediate from the form of the matrices. □

10.7. Some tests. Let us make several remarks on Conjecture 1, which we are going to prove in dimension 3 by a case analysis.

First we analyze trees, and for each marked tree we complete it according to Theorem 5. We use systematically Theorem 16 and also we may sometimes use the following simple

Remark 10.8 (Minimality). If setting two or more of the variables ξ_i equal we have an irreducible polynomial then the one we started with is irreducible.

Lemma 10.9 (Parity test). *i) If we compute t at an odd number g , the determinant $\chi_{\mathcal{A}}(g) \neq 0$.*

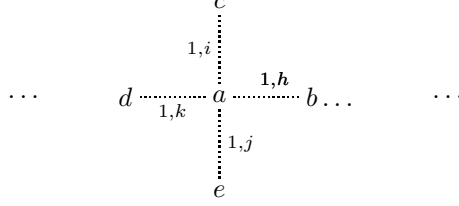
ii) If a linear form $t + \sum_i a_i \xi_i$, $a_i \in \mathbb{Z}$ divides $\chi_{\mathcal{A}}(t)$ we must have $\sum_i a_i$ is even.

Proof. i) We compute modulo 2 and set all $\xi_i = 1$, recalling that $L_a(\xi)|_{\xi_i=1} = 0, 2$. We get $\chi_{\mathcal{A}}(t) = t^m$.

ii) A linear form $t + \sum_i a_i \xi_i$, $a_i \in \mathbb{Z}$ divides $\chi_{\mathcal{A}}(t)$ if and only if we have $\chi_{\mathcal{A}}(-\sum_i a_i \xi_i) = 0$, then set $\xi_i = 1$ and use the first part. □

We shall use the parity test as follows.

Theorem 17. *Suppose we have a graph A , in which we find a vertex a and an index, say 1 , so that*



we have:

- 1 appears in all and only the edges having a as vertex.
- When we remove a (and the edges meeting a) we have a connected graph A with at least 2 vertices.
- When we remove the edges associated to any index, the factors described in Theorem 16 are irreducible.

Then the polynomial associated to A is irreducible.

Proof. We take a as root, setting $\xi_1 = 0$ we have by Theorem 16 and the hypotheses, that $\chi_A(t) = tP(t)$ with $P = \chi_A(t)$ irreducible of degree > 1 . Thus, if the polynomial $\chi_A(t)$ factors, then it must factor into a linear $t - \ell(\xi)$ times an irreducible polynomial of degree > 1 .

Moreover modulo $\xi_1 = 0$ we have that 0 and ℓ coincide, thus ℓ is a multiple of ξ_1 .

Take another index $i \neq 1, h$ if a is an end and the only edge from a is marked $(1, h)$ otherwise just different from 1 and set $\xi_i = 0$. Now the polynomial $\chi_A(t)$ specializes to the product $\prod_j \chi_{A_j}(t)$ where the A_j are the connected component of the graph obtained from A by removing all edges in which i appears as marking. By hypothesis $\{a\}$ is not one of the A_j .

If no factor is linear we are done. Otherwise there is an isolated vertex $d \neq a$ so that $\{d\}$ is one of the connected components A_j . The linear factor associated is $t - \sigma(d)L_d(\xi)|_{\xi_i=0}$. Clearly we have that the coefficient of ξ_1 in $L_d(\xi)$ is ± 1 (since the marking 1 appears only once). This implies that $\ell = t \pm \xi_1$ and this is not possible by the parity test. \square

This Theorem can be used as the basis of a possible, induction. Let us analyze this.

Take a graph A of dimension and rank n . Thus it has $n + 1$ vertices. Assume that, by some inductive procedure, we know that, if we remove the edges associated to any index, the factors described in Theorem 16 are irreducible. If there is an index i such that, when we remove the edges containing i the graph remains connected, we are done. Similarly if we can find an index satisfying the conditions of Theorem 17.

11. DIMENSION 3

11.1. Bases and encoding graphs. We first classify the graphs by rank and up to the symmetry induced by permuting the variables e_i (that is the action of the symmetric group S_m).

Thus in order to classify these graphs, the first step is to understand the possible lists ν_i which produce a connected complete graph.

A choice of a maximal tree in each such graph determines, through its edges, a linearly independent list extracted from the vectors $E := \{e_i \pm e_j\}$. Thus we may start by first classify up to symmetry such lists of rank k . In dimension n will appear lists of rank $k \leq n$.

Since we are trying to classify the graphs up to equivalence we have some freedom in choosing the root. In a tree with n vertices we can always choose a vertex r as root so that every other vertex is at distance at most $\lceil n/2 \rceil$, thus each possible ν_h is obtained by adding up at most $\lceil n/2 \rceil$ elements in the list $e_i \pm e_j$.

This gives a finite algorithm which is still computationally very heavy even in dimension 3.

Let us start by explaining how to classify the lists of independent edges for dimension 3.

Observe first that $E := \{e_i \pm e_j\}$ decomposes under the symmetric group into 2 orbits, the *black* and *red* edges.

In a list it is first convenient to count the two numbers e, f of black and red edges, and we may have for a list with 3 elements 4 possibilities $(3, 0), (2, 1), (1, 2), (0, 3)$. It is convenient to display the list by its *encoding graph*. This is the subgraph of the full graph with vertices the numbers $1, \dots, m$ formed by the edges (i, j) which appear as markings in the graph A . We can also color and orient these edges.

When we take 3 linearly independent markings, their encoding graph up to permutation of the variables can have the following different combinatorial structure

$$\text{Type 0 : } \begin{array}{ccc} 2 & 4 & 6 \\ | & | & | \\ 1 & 3 & 5 \end{array} , \quad (1, 2), (3, 4), (5, 6)$$

$$\text{Type 1 : } \begin{array}{ccc} & 1 & 5 \\ & | & | \\ 3 & \text{---} 2 & 4 \end{array} , \quad (1, 2), (2, 3), (4, 5)$$

$$\text{Type 2 : } \begin{array}{ccc} & 1 & \\ & | & \\ 3 & \text{---} 2 & \text{---} 4 \end{array} , \quad (1, 2), (2, 3), (1, 4)$$

$$\text{Type 3 : } 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4 , \quad (1, 2), (2, 3), (3, 4)$$

In all these cases we may have any choice of red edges.

We now discuss two special cases.

Type A : We may have at the same time an edge, which we may assume to be $(1, 2)$ black and red, plus another edge, in general disjoint $(3, 4)$ but after specialization it can be assumed to be $(2, 3)$ black or red.

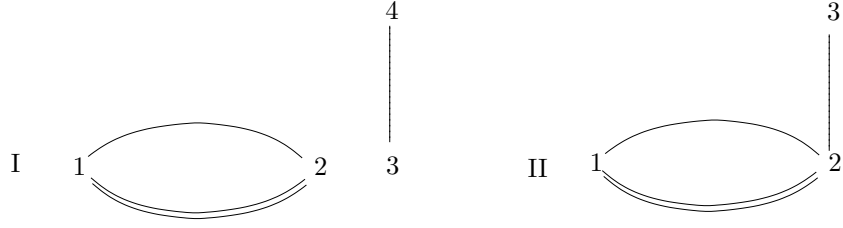
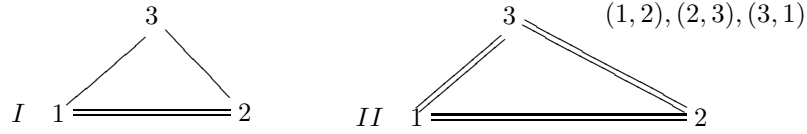


FIGURE 2. The dotted line may be either red or black.

Finally we may have an *odd circuit* (notice that an even circuit gives linearly dependent markings)

(89)

Type B :



Type BII cannot occur since, if we have two adjacent red edges with a common index in the label, this configuration has to be completed with a black edge.

In case BI the encoding graph is completely described by the choice of orientation for the black edges (and again we have some symmetries to consider).

It is clear that *specializing some variables* one can pass from more general types to others. When we pass to analyzing the possible allowable graphs, the 3 markings come from a maximal tree and the actual graph may need to be completed with further edges.

11.2. Irreducibility tests.

There are standard algorithms that check if a polynomial with integer coefficients is irreducible. In our analysis we try to avoid them as much as possible in order to give a possible general approach, nevertheless for a few cases we could not find a theoretical explanation so we just verified the irreducibility with these algorithms.

If A has s vertices the characteristic polynomial is of degree s . For the markings, we restrict to the case of rank $s - 1$ (by Theorem 12). Let us start by considering:

$s = 2$. In this case A is a single edge that we may assume marked by $(1, 2)$ or equivalently $(2, 1)$. The corresponding matrix is, in the black and red case:

$$\begin{vmatrix} t & -2\sqrt{\xi_1\xi_2} \\ -2\sqrt{\xi_1\xi_2} & t + \xi_1 - \xi_2 \end{vmatrix}, \quad \begin{vmatrix} t & 2\sqrt{\xi_1\xi_2} \\ -2\sqrt{\xi_1\xi_2} & t + \xi_1 + \xi_2 \end{vmatrix}$$

with determinants

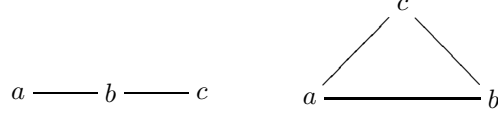
$$(90) \quad t^2 + t(\xi_1 - \xi_2) - 4\xi_1\xi_2, \quad t^2 + t(\xi_1 + \xi_2) + 4\xi_1\xi_2$$

it is easily seen that these polynomials in t, ξ_1, ξ_2 are irreducible. $s = 3$.

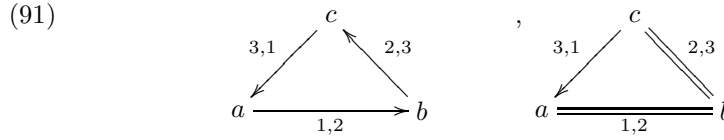
Lemma 11.3. *Unless we are in the special component we cannot have in the graph adjacent $a \xrightarrow{1,2} b \xrightarrow{1,2} c$ or $a \xrightarrow{1,2} b \xrightarrow{2,1} c$.*

Proof. We use the fact that $(H_{1,2} \cup H_{2,1}) \cap S_{1,2} = \{v_1, v_2\}$. \square

$s = 3$. For a graph A with 3 vertices $\chi_A(t)$ has degree 3, we only have two possibilities, up to coloring the edges.



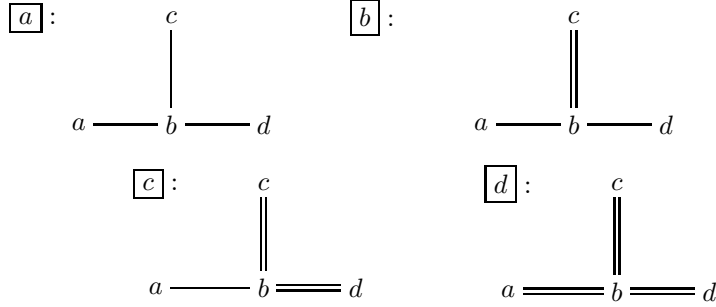
We may have, once we specialize the variables, only the case $(1,2), (2,3)$ on the tree with various colors and orientations. Some of these choices do not give a complete graph. We complete it (this determines uniquely the oriented edges to be added) we have the circuit which must be even by the rank assumption and up to symmetry



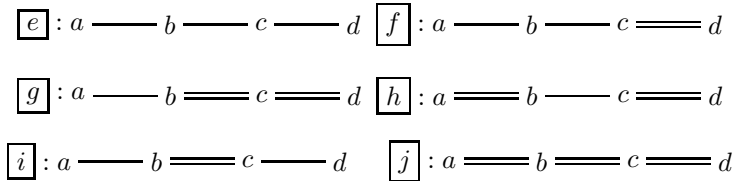
All these cases fall under the requirements of Theorem 17 and thus give irreducible polynomials.

$s = 4$ The possible colored trees with 4 vertices are

• **Star types**



• **Linear types**



For each of these trees we must analyze the markings which can be classified according to their encoding graphs. We easily see that, for the encoding graphs of types 0,1,2,3 there is always an index of a marking which appears only once and in an end edge. Thus all these cases fall under Theorem 17.

Type A cannot occur in the star type and in cases e, h,i,j by Lemma 11.3.

The analysis is still quite long. We can simplify it a bit if we weaken our requirements and try to prove only that the polynomials are irreducible over \mathbb{Z} . This is in any case

sufficient for the applications we have in mind. We can set all variables $\xi_i = \frac{y}{\sqrt{2}}$. In fact when we do this for types a through d we always have a linear by an irreducible cubic polynomial. This is not a useful information but for the other cases we have some reductions. For the graphs of type e the matrix becomes

$$\begin{vmatrix} t & -y & 0 & 0 \\ -y & t & -y & 0 \\ 0 & -y & t & -y \\ 0 & 0 & -y & t \end{vmatrix}$$

with determinant

$$t^4 - 3t^2y^2 + y^4 = (t^2 - ty - y^2)(t^2 + ty - y^2)$$

For colored case we set $\xi_i = y$ and we have as possible matrices

$$\boxed{f} \quad a \text{ --- } b \text{ --- } c \text{ === } d^- , \quad \begin{vmatrix} t & -2y & 0 & 0 \\ -2y & t & -2y & 0 \\ 0 & -2y & t & 2y \\ 0 & 0 & -2y & t+2y \end{vmatrix}$$

with determinant

$$t^4 + 2t^3y - 4t^2y^2 - 16ty^3 - 16y^4 = (t+2y)(t^3 - 4ty^2 - 8y^3)$$

$$\boxed{g} \quad a \text{ --- } b \text{ === } c^- \text{ === } d^+ , \quad \begin{vmatrix} t & -2y & 0 & 0 \\ -2y & t & 2y & 0 \\ 0 & -2y & t+2y & -2y \\ 0 & 0 & 2y & t \end{vmatrix}$$

with determinant the irreducible polynomial

$$t^4 + 2t^3y + 4t^2y^2 - 8ty^3 - 16y^4$$

$$\boxed{h} \quad a \text{ === } b^- \text{ --- } c^- \text{ === } d^+ , \quad \begin{vmatrix} t & 2y & 0 & 0 \\ -2y & t+2y & 2y & 0 \\ 0 & 2y & t+2y & -2y \\ 0 & 0 & 2y & t \end{vmatrix}$$

with determinant the irreducible polynomial

$$t^4 + 2t^3y + 4t^2y^2 + 8ty^3 + 16y^4$$

$$\boxed{i} \quad a \text{ --- } b^+ \text{ === } c^- \text{ --- } d^- , \quad \begin{vmatrix} t & -2y & 0 & 0 \\ -2y & t & 2y & 0 \\ 0 & -2y & t+2y & 2y \\ 0 & 0 & 2y & t+2y \end{vmatrix}$$

with determinant the polynomial

$$t^4 + 4t^3y - 8ty^3 = t(t+2y)(t^2 + 2ty - 4y^2)$$

$$\boxed{j} \quad a \text{ === } b^- \text{ === } c^+ \text{ === } d^- , \quad \begin{vmatrix} t & 2y & 0 & 0 \\ -2y & t+2y & -2y & 0 \\ 0 & 2y & t & 2y \\ 0 & 0 & -2y & t+2y \end{vmatrix}$$

with determinant the irreducible polynomial

$$t^4 + 4t^3y + 16t^2y^2 + 24ty^3 + 16y^4$$

this shows that,

- Lemma 11.4.**
- If for a given marking of the tree of type e the polynomial factors it must be into two irreducible quadratic polynomials.
 - If for a given marking of the tree of type f the polynomial factors it must be into a linear and an irreducible cubic polynomials.
 - Types g, j, h are always irreducible.
 - For type i we have no useful information.

Recall the classification of §11.1. So we are left with Linear types of type A in case f.g. Type BI for trees of linear and star type.

Linear types Type A. In the linear graphs 3 must be in the middle so if the polynomial factors it must be into 2 irreducible quadratics. We have already seen that we only need to treat cases f,g and for these the previous factorization is incompatible with the restrictions given by Lemma 11.4.

We still need to treat Type B I which can appear only in the two cases f, i . Some orientations give rise to a graph in the special component so we may ignore them. We are left with (up to exchanging 1, 2)

$$\begin{aligned} \boxed{f1} \quad & a \xrightarrow{3,1} b^+ \xrightarrow{2,3} c^+ \xrightarrow{1,2} d^- , \quad \boxed{f2} \quad a \xrightarrow{3,1} b^+ \xrightarrow{3,2} c^+ \xrightarrow{1,2} d^- , \\ \boxed{f3} \quad & a \xrightarrow{1,3} b^+ \xrightarrow{2,3} c^+ \xrightarrow{1,2} d^- , \quad \boxed{f4} \quad a \xrightarrow{1,3} b^+ \xrightarrow{3,2} c^+ \xrightarrow{1,2} d^- , \end{aligned}$$

Now, $f1, f4$ do not arise in the applications, since they are not complete (although they also give rise to irreducible polynomials). We are left with $f2, f4$:

$$\begin{aligned} f2 = & \begin{vmatrix} t & -2\sqrt{\xi_1\xi_3} & 0 & 0 \\ -2\sqrt{\xi_1\xi_3} & t - \xi_1 + \xi_3 & -2\sqrt{\xi_2\xi_3} & 0 \\ 0 & -2\sqrt{\xi_2\xi_3} & t - \xi_1 - \xi_2 + 2\xi_3 & 2\sqrt{\xi_2\xi_1} \\ 0 & 0 & -2\sqrt{\xi_2\xi_1} & t + 2\xi_3 \end{vmatrix} \\ f3 = & \begin{vmatrix} t & -2\sqrt{\xi_1\xi_3} & 0 & 0 \\ -2\sqrt{\xi_1\xi_3} & t - \xi_3 + \xi_1 & -2\sqrt{\xi_2\xi_3} & 0 \\ 0 & -2\sqrt{\xi_2\xi_3} & t - 2\xi_3 + \xi_1 + \xi_2 & 2\sqrt{\xi_2\xi_1} \\ 0 & 0 & -2\sqrt{\xi_2\xi_1} & t + 2\xi_3 \end{vmatrix} \end{aligned}$$

Markings

$$0, \xi_1 - \xi_3, \xi_1 + \xi_2 - 2\xi_3, 2\xi_3; \quad 0, \xi_3 - \xi_1, 2\xi_3 - \xi_1 - \xi_2, 2\xi_3.$$

In this table we show the possible linear factors setting one of the variables to 0.

| | $f2$ | $f3$ |
|---------------|---------------------------|--------------------------|
| $\xi_1 = 0 :$ | $0, 2\xi_3$ | $0, 2\xi_3$ |
| $\xi_2 = 0 :$ | $-\xi_1 + 2\xi_3, 2\xi_3$ | $\xi_1 - 2\xi_3, 2\xi_3$ |
| $\xi_3 = 0 :$ | $0, -\xi_1$ | $0, \xi_1$ |

So a possible linear factor could be $t - \xi_1 + 2\xi_3, t + 2\xi_3$ in the first case, $t + 2\xi_3, t + \xi_1 - 2\xi_3$ in the second. The odd cases can be excluded by parity so we may have $t + 2\xi_3$ in both cases. At $t = -2\xi_3$ the two determinants are

$$-8\xi_1\xi_2(\xi_1 - \xi_3)\xi_3, \quad -24\xi_1\xi_2(\xi_1 - \xi_3)\xi_3.$$

Finally Type i , observe that

$$a \xrightarrow{3,1} b \xrightarrow{1,2} c \xrightarrow{2,3} d, \quad a \xrightarrow{1,3} b^+ \xrightarrow{1,2} c^- \xrightarrow{2,3} d^-$$

$$a \xrightarrow{3,1} b \xrightarrow{1,2} c \xrightarrow{3,2} d,$$

do not arise since they are not complete. It remains up to symmetry

$$a \xrightarrow{1,3} b^+ \xrightarrow{1,2} c^- \xrightarrow{3,2} d^- ,$$

$$\begin{vmatrix} t & -2\sqrt{\xi_1\xi_3} & 0 & 0 \\ 2\sqrt{\xi_1\xi_3} & t - \xi_3 + \xi_1 & 2\sqrt{\xi_2\xi_1} & 0 \\ 0 & -2\sqrt{\xi_2\xi_1} & t + 2\xi_1 + \xi_2 - \xi_3 & 2\sqrt{\xi_2\xi_3} \\ 0 & 0 & 2\sqrt{\xi_2\xi_3} & t + 2\xi_1 + 2\xi_2 - 2\xi_3 \end{vmatrix}$$

This case has passed a standard factorization algorithm which shows that it is irreducible.

Star types

Type B I For only one orientation the tree is complete. Take b as root

$$\begin{array}{ccc} c^+ & \text{markings} & \xi_3 - \xi_2 \\ \begin{array}{c} 3,2 \\ | \\ a^- \xrightarrow{1,2} b \xrightarrow{3,1} d^+ \end{array} & & \begin{array}{c} \xi_2 + \xi_1 \text{ --- } | \text{ --- } \xi_3 - \xi_1 \end{array} \end{array}$$

$$\begin{vmatrix} t + \xi_2 + \xi_1 & -2\sqrt{\xi_1\xi_2} & 0 & 0 \\ 2\sqrt{\xi_1\xi_2} & t & -2\sqrt{\xi_2\xi_3} & -2\sqrt{\xi_1\xi_3} \\ 0 & -2\sqrt{\xi_2\xi_3} & t - \xi_2 + \xi_3 & 0 \\ 0 & -2\sqrt{\xi_1\xi_3} & 0 & t + \xi_3 - \xi_1 \end{vmatrix}$$

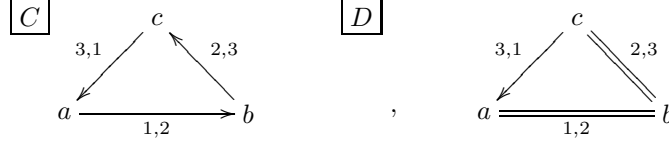
passes standard tests of irreducibility.

In fact in this case even the determinant is irreducible (over \mathbb{Q}):

$$4 \left(\xi_1^2 \xi_2^2 + \xi_1^2 \xi_2 \xi_3 + \xi_1 \xi_2^2 \xi_3 - \xi_1^2 \xi_3^2 - \xi_1 \xi_2 \xi_3^2 - \xi_2^2 \xi_3^2 \right),$$

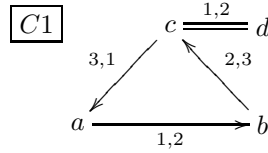
Circuits

We first start from the Formulas (91) of a basic circuit

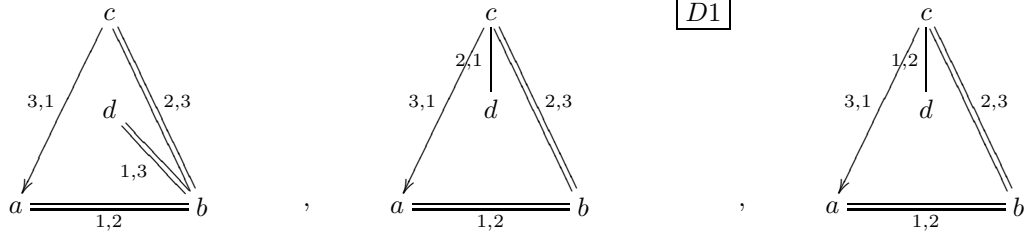


and add a linearly independent edge in all possible ways and then close it as necessary. If this edge has two indices disjoint from 1, 2, 3 we are under the hypotheses of Theorem 17. Otherwise, up to symmetry we may have several cases, but we can always assume that 1 appears in the marking of the edge. It can be (1, 4), black or red, in this case, we complete if necessary the graph but then we may apply Theorem 17. The other possibility is that it does not involve an index different from 1, 2, 3 but then it must have a color different from the one appearing in the graph in order to be linearly independent.

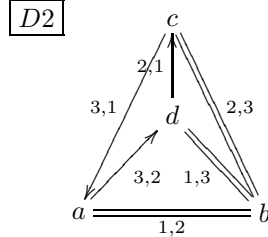
Type C We may assume we add the edge $\overline{\overline{1,2}}$, according to Lemma 11.3 the only possible position is at c .



Type D We may assume we add the edge $\overline{\overline{1,3}}$ or $\overline{\overline{1,2}}$, according to Lemma 11.3 the only possible positions are



the first 2 should be completed to the same



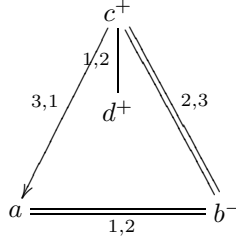
So our final computation is with these 3 cases. The matrices are:

$$\boxed{C1} \begin{vmatrix} t & -2\sqrt{\xi_1\xi_2} & -2\sqrt{\xi_1\xi_3} & 0 \\ -2\sqrt{\xi_1\xi_2} & t + \xi_1 - \xi_2 & -2\sqrt{\xi_2\xi_3} & 0 \\ -2\sqrt{\xi_1\xi_3} & -2\sqrt{\xi_2\xi_3} & t + \xi_1 - \xi_3 & 2\sqrt{\xi_1\xi_2} \\ 0 & 0 & -2\sqrt{\xi_1\xi_2} & t + 2\xi_1 + \xi_2 - \xi_3 \end{vmatrix}$$

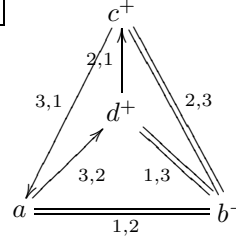
passes a standard test of irreducibility. The determinant is ξ_1 times an irreducible cubic.

Take a as root.

$\boxed{D1}$



$\boxed{D2}$

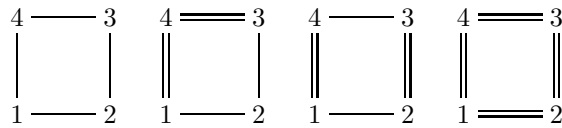


$$\begin{vmatrix} t & 2\sqrt{\xi_1\xi_2} & -2\sqrt{\xi_1\xi_3} & 0 \\ -2\sqrt{\xi_1\xi_2} & t + \xi_1 + \xi_2 & -2\sqrt{\xi_2\xi_3} & 0 \\ -2\sqrt{\xi_1\xi_3} & 2\sqrt{\xi_2\xi_3} & t + \xi_1 - \xi_3 & -2\sqrt{\xi_1\xi_2} \\ 0 & 0 & -2\sqrt{\xi_1\xi_2} & t + 2\xi_1 - \xi_3 + \xi_2 \end{vmatrix}$$

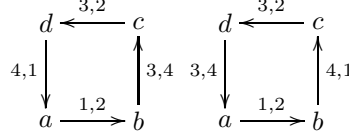
$$\begin{vmatrix} t & 2\sqrt{\xi_1\xi_2} & -2\sqrt{\xi_1\xi_3} & -2\sqrt{\xi_2\xi_3} \\ -2\sqrt{\xi_1\xi_2} & t + \xi_1 + \xi_2 & -2\sqrt{\xi_2\xi_3} & -2\sqrt{\xi_1\xi_3} \\ -2\sqrt{\xi_1\xi_3} & 2\sqrt{\xi_2\xi_3} & t + \xi_1 - \xi_3 & -2\sqrt{\xi_1\xi_2} \\ -2\sqrt{\xi_2\xi_3} & 2\sqrt{\xi_1\xi_3} & -2\sqrt{\xi_1\xi_2} & t + \xi_3 - \xi_2 \end{vmatrix}$$

In the first the characteristic polynomial is irreducible while the determinant is $4\xi_1$ times an irreducible cubic. In the second they are both irreducible.

We finally have circuits with 4 edges but not 3. In this case the circuit has 4 markings linearly dependent of rank 3, which satisfy a very special linear relation. One easily sees that they must have as encoding graph a standard circuit with an even number of red edges and suitably oriented black edges:



One easily sees that they fall under theorem 17. E.g.:



The other cases are similar or not complete.

12. SEPARATION

12.1. Translations. We want to prove here the separation Lemma 6.17 in dimension 3.

We have constructed the polynomials which appear in the formulas for $ad(N)$ as follows.

We take a set $A = \{(a_0, \sigma_0), (a_1, \sigma_1), \dots, (a_k, \sigma_k)\} \subset \mathbb{Z}^m \times \mathbb{Z}/(2)$ so that the vectors a_i are independent in the sense of affine geometry (they span an affine space of dimension k).

We only consider those such that the complete graph generated by them is connected and we call k the *rank* of A . Let us call these sets *allowable*.

To such a set we have associated (Definition 10.2) a $k+1 \times k+1$ matrix C_A and its characteristic polynomial χ_A , a polynomial of degree $k+1$ in t and the variables ξ_i which is monic in t . We Conjecture that

Conjecture 2 If $A \neq B$ have at least 2 elements then $\chi_A \neq \chi_B$.

For one element $(a, +)$ and $(-a, -)$ give $t - L_a(\xi)$.

We start from a standard edge $[(0, +), (e_1 - e_2, +)]; [(0, +), (-e_1 - e_2, -)]$.

Under translations and sign change we have 4 possibilities:

$$\begin{aligned}
 &[(a, +), (a + e_1 - e_2, +)]; \quad [(b, +), (-b - e_1 - e_2, -)]. \\
 &[(c, -), (c + e_1 - e_2, -)]; \quad [(d, -), (-d - e_1 - e_2, +)].
 \end{aligned}$$

The matrices are

$$\begin{vmatrix} -a(\xi) & 2\sqrt{\xi_1 \xi_2} \\ 2\sqrt{\xi_1 \xi_2} & -a(\xi) - \xi_1 + \xi_2 \end{vmatrix}, \quad \begin{vmatrix} -b(\xi) & -2\sqrt{\xi_1 \xi_2} \\ 2\sqrt{\xi_1 \xi_2} & -b(\xi) - \xi_1 - \xi_2 \end{vmatrix} \\
 \begin{vmatrix} c(\xi) & -2\sqrt{\xi_1 \xi_2} \\ -2\sqrt{\xi_1 \xi_2} & c(\xi) + \xi_1 - \xi_2 \end{vmatrix}, \quad \begin{vmatrix} d(\xi) & 2\sqrt{\xi_1 \xi_2} \\ -2\sqrt{\xi_1 \xi_2} & d(\xi) + \xi_1 + \xi_2 \end{vmatrix}$$

Lemma 12.2. *These blocks can be reconstructed from the characteristic polynomial.*

Proof. First, passing modulo 2 we identify from the trace τ the two variables ξ_1, ξ_2 . We can deduce the elements a, b, c, d in the 4 cases from τ .

$$-\tau = 2(a - \xi_2) + \xi_1 + \xi_2 = 2b + \xi_1 + \xi_2 = 2c + \xi_2 - \xi_1 = 2d - \xi_1 - \xi_2.$$

So let us write $\tau = -2a + \xi_2 - \xi_1$ and the possible 4 matrices with trace τ are

$$\begin{vmatrix} -a & 2\sqrt{\xi_1 \xi_2} \\ 2\sqrt{\xi_1 \xi_2} & -a - \xi_1 + \xi_2 \end{vmatrix}, \quad \text{or} \quad \begin{vmatrix} -a + \xi_2 & -2\sqrt{\xi_1 \xi_2} \\ 2\sqrt{\xi_1 \xi_2} & -a - \xi_1 \end{vmatrix} \\
 \begin{vmatrix} a + \xi_1 - \xi_2 & -2\sqrt{\xi_1 \xi_2} \\ -2\sqrt{\xi_1 \xi_2} & a + 2\xi_1 - 2\xi_2 \end{vmatrix}, \quad \begin{vmatrix} a + \xi_1 & 2\sqrt{\xi_1 \xi_2} \\ -2\sqrt{\xi_1 \xi_2} & a + 2\xi_1 + \xi_2 \end{vmatrix}$$

we compute the 4 determinants

$$\begin{aligned} & a(a + \xi_1 - \xi_2) - 4\xi_1\xi_2, \quad (a - \xi_2)(a + \xi_1) + 4\xi_1\xi_2, \\ & (a + \xi_1 - \xi_2)(a + 2\xi_1 - 2\xi_2) - 4\xi_1\xi_2, \quad (a + \xi_1)(a + 2\xi_1 + \xi_2) + 4\xi_1\xi_2. \end{aligned}$$

We leave to the reader to verify that, when they are equal, the edge is the same.⁴

□

Proof In dimension 2,3. We have already proved in Lemma 12.2, that this statement is true for blocks with two elements. In dimension 3 we know that all polynomials χ_C with C of rank < 4 are irreducible.

Consider thus the m linear mappings $\lambda_i : \mathbb{Z}^m \rightarrow \mathbb{Z}^{m-1}$ each dropping the i^{th} coordinate or in more intuitive terms, setting $e_i = 0$.

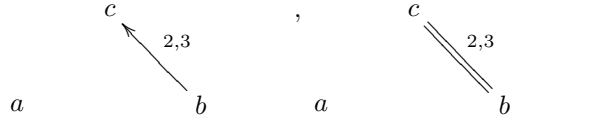
We extend this map to $\lambda_i : \mathbb{Z}^m \times \mathbb{Z}/(2) \rightarrow \mathbb{Z}^{m-1} \times \mathbb{Z}/(2)$ and notice that, if we remove from the graph $\mathbb{Z}^m \times \mathbb{Z}/(2)$ all edges in which e_i appears, this is a map of graphs. Moreover take A allowable of rank k and remove all the edges in which e_i appears, we obtain in general several connected components A_1^i, \dots, A_s^i of the restricted graph. It is easily seen that every connected component B of the induced graph maps injectively under λ_i to $\mathbb{Z}^{m-1} \times \mathbb{Z}/(2)$, since the index i does not appear in the markings of the edges of B . Theorem 16 tells us that

$$(92) \quad \chi_A(t, \xi)_{\xi_i=0} = \prod_{j=1}^s \chi_{A_j^i}(t, \xi).$$

This is again the basis of an induction. We can in fact by induction reconstruct the entire graph once we forget the i^{th} coordinate, except for the isolated vertices for which we have no information on the sign. By convention we describe their known coordinates as if the sign were $+$.

In particular by setting all variables equal to 0 except two of them we see that if $\chi_A = \chi_B$ the two graphs must have the same colored encoding graphs. In fact we recover not only the edges but also the coordinates of the vertices relative to the two indices of the edge and their sign.

Now in dimension 3 we treat first the case of a block with 3 vertices. Thus we know the encoding graph. This has either two or three edges. Let us treat the case of 3 edges (the other case is simpler). In this case the graph is as in (91), we need to compute the coordinates. In this case set for instance $e_1 = 0$ we see



In the first case we know the sign of a and hence the second and third coordinate of all the vertices. This is enough to reconstruct all edges since b, c have different third coordinate.

In the second case, assume for instance c has sign $+$. We see immediately to which of b, c the point a is connected by the black edge $(1, 3)$ since we must have one and only one $a_2 = c_2$ or $a_2 = -b_2 = c_2 + 1$.

Next the most degenerate cases with 4 vertices.

This, up to symmetry is when the encoding graph is of type A, B (cf. 11.1).

⁴of course the matrix depends on ordering the vertices of the edge, in particular one vertex appears as root

In Type A when we put $e_3 = 0$ we must see the graph

$$a \xrightarrow{1,2} b \quad c \xrightarrow{1,2} d$$

together with the first two coordinates and the sign of all the vertices.

Set $a = (a_1, a_2, x), c = (c_1, c_2, y)$ so that

$$(93) \quad b = (-1 - a_1, -1 - a_2, -x), \quad d = (c_1 + 1, c_2 - 1, y).$$

When we set $e_2 = 0$ we have two possible cases:

$$1) \quad u \xrightarrow{1,3} v \quad z \quad w$$

or

$$2) \quad u \xrightarrow{1,3} v \quad z \quad w.$$

Possibly we may have $z = w$. Let us treat the first case.

We need to determine which vertices match coordinates and are thus joined by this edge. If say $a = (a_1, a_2, x)$ is joined with $c = (c_1, c_2, y)$ then we must have that $a + c = (-1, 0, -1)$ so

$$a_1 + c_1 = -1, a_2 + c_2 = 0, x + y = -1.$$

Similarly for the other 3 possible ways in which the edge may appear. We need to show that only one possible choice is available, so we may assume that the choice a, c is available and then prove that the others are not possible. Let us first exclude the possibility b, c . By (93) $b_2 + c_2 = -1 - a_2 + c_2$ by assumption. If a, c is available we have also $a_2 + c_2 = 0$, if also b, c is available then $0 = b_2 + c_2 = -1 - a_2 + c_2$. This implies $c_2 = 1/2$ which is not possible.

Now the possibility b, d or a, d . We have that $d = (c_1 + 1, c_2 - 1, y)$ so for a, d again we should have $a_2 + d_2 = 0$ that is $a_2 + c_2 - 1 = 0$ incompatible. For b, d we would have

$$b_1 + d_1 = -1, b_2 + d_2 = 0, -x + y = -1.$$

We have $y = -1, x = 0$ and $b_1 + d_1 = a_1 + c_1 = -1$. Next $b_1 + d_1 = -1 - a_1 + c_1 + 1$ which implies $a_1 = 0, c_1 = -1 - a_1 + c_1 = 1$ hence $c_1 = -1, a_1 = 0$ then $b_2 + d_2 = a_2 + c_2 = 0$ implies $-1 - a_2 + c_2 - 1 = 0$ that is $a_2 = -1, c_2 = 1$. Finally we must have

$$\begin{array}{ccc} (0, -1, 0) & \xrightarrow{1,2} & (-1, 0, 0) \\ \parallel_{1,3} & & \parallel_{1,3} \\ (-1, 1, -1) & \xrightarrow{1,2} & (0, 0, -1) \end{array}.$$

This is possible only if we ignore the signs. With signs this is incompatible (the odd circuits do not lift from $\tilde{\Lambda}$ to Λ and hence we have excluded them with the resonance hypothesis 3) 9.1).

Now the second case.

When we put $e_3 = 0$ we must see the graph

$$a \xrightarrow{1,2} b \quad c \xrightarrow{1,2} d.$$

When we set $e_2 = 0$ we have a graph

$$u \xrightarrow{1,3} v \quad z \quad w$$

We need to determine which vertices are joined by this edge. If say $a = (a_1, a_2, x)$ is joined with $c = (c_1, c_2, y)$, we have still two possibilities for the orientation. In have have either $a - c = (-1, 0, 1)$, $a - c = (1, 0, -1)$ so

$$i) \quad a_1 - c_1 = -1, a_2 = c_2, x - y = 1 \quad \text{or} \quad ii) \quad c_1 - a_1 = -1, a_2 = c_2, y - x = 1.$$

Either u or v correspond to a , that is either $u_1 = a_1$ or $v_1 = a_1$ moreover we need to have that b, d correspond to z, w in some order. Similarly for the other 3 possible ways in which the edge may appear. We need to show that only one possible choice is available, so we may assume that the choice a, c is available and then prove that the others are not possible. Let us first exclude the possibility b, c . This implies, $b_2 = c_2 = a_2$ since $b = (-1 - a_1, -1 - a_2, -x)$ we have $a_2 = -1/2$ impossible.

We have $d_1 = c_1 + 1, d_2 = c_2 - 1, d_3 = y$. Assume we have the edge b, d marked $(1, 3)$.

$$j) \quad b_1 - d_1 = 1, b_2 = d_2, -x - y = -1 \quad \text{or} \quad jj) \quad d_1 - b_1 = 1, d_2 = b_2, y + x = -1.$$

Assume case i, j . Then $b_2 = d_2 = c_2 - 1$ so $-1 - a_2 = a_2 - 1$ implies $a_2 = 0$. We get

$$a = (-2, 0, 1), \quad b = (1, -1, -1), \quad c = (-1, 0, 0), \quad d = (0, -1, 0).$$

$$\begin{array}{ccc} (0, -1, 0) & \xrightarrow{2,1} & (-1, 0, 0) \\ \begin{array}{c} 3,1 \\ \downarrow \end{array} & & \begin{array}{c} 1,3 \\ \downarrow \end{array} \\ (1, -1, -1) & \xrightarrow{1,2} & (-2, 0, 1) \end{array}.$$

This is possible only if we ignore the signs. The other cases are symmetric due to the symmetry in the red edge $(1, 2)$.

A priori there is a possible further degeneracy, this occurs when the two edges

$$a \xrightarrow{1,2} b \quad c \xrightarrow{1,2} d$$

coincide in the projection. This means that

$$a + b = -e_1 - e_2, \quad a - b = \pm(e_2 - e_1) \implies a = d = -e_1, b = c = -e_2 \text{ or conversely.}$$

Then the original points can be

$$a = (-1, 0, x), b = (0, -1, -x); \quad c = (0, -1, y), d = (-1, 0, y).$$

But we cannot have two of them joined by $(1, 3)$ so this case does not occur.

In Type *BI* (89), when we put $e_3 = 0$ we must see the graph

$$a \xrightarrow{1,2} b \quad c \quad d.$$

We have then $a = (a_1, a_2, x), b = (-1 - a_1, -1 - a_2, -x)$ and we know the sign of both. When we set $e_2 = 0$ we have a graph

$$u \xrightarrow{1,3} v \quad z \quad w$$

When we set $e_1 = 0$

$$p \xrightarrow{2,3} q \quad t \quad s$$

We need to determine which vertices are joined by these edges. What we know are the first two coordinates and sign of a, b , the last two coordinates and sign of the vertices p, q the first and last of u, v .

If the signs of u, v and p, q are different then we see immediately that the graph is uniquely reconstructed. Let us give some detail, for instance assume that u, v have the same sign of a and p, q the sign of b . Then necessarily either $u = a$ or $v = a$ and which one is the case we see by the first coordinate. Similarly for p, q .

If u, v, p, q have the same sign then the two edges (u, v) and (p, q) form a segment. This a priori can occur in 4 ways, but two of them are not complete. The only possibilities are

$$u \xrightarrow{1,3} v = p \xrightarrow{2,3} q \qquad p \xrightarrow{2,3} q = u \xrightarrow{1,3} v$$

Which one occurs is determined inspecting the coordinates (the two end points have different third coordinate).

Next we need to identify in this segment the point a or b depending from the sign, and this is clear again by the coordinates.

Next cases C1, D1, D2 are distinguished by their encoding graphs. In case C1 we set $e_3 = 0$ and see the two edges $(1, 2)$. How to complete the graph is clear by inspecting the second coordinate since we know all the signs.

Cases D1, D2 are treated in a very similar way.

The less degenerate cases are easier since we have more coordinates available that change and follow the same line of reasoning. \square

13. REAL ROOTS

In this section we want to touch on the issue of when the eigenvalues of the combinatorial matrices are all real. We know that, for black edges and positive ξ the quadratic form is positive definite so the corresponding matrix has real eigenvalues. When we have red edges we do not have always real eigenvalues and we need to isolate the regions in the parameters where this occurs.

First of all in dimension 2 we have seen that for a 2 block corresponding to a red edge marked (i, j) we have the inequality $(\xi_i + \xi_j)^2 > 16\xi_i\xi_j$ determining the region where the eigenvalues are real.

If one follows the choice of [10] no further blocks appear and these inequalities suffice and determine a non empty open sector in the parameters ξ .

Otherwise we have to analyze the 3 dimensional blocks.

Similarly in dimension 3. The polynomial inequalities that one obtains are explicit but rather formidable, so we can discuss just the qualitative aspects. One can remark that when we set one variable equal to 0 we are in position to apply Theorem 16. When we have linear or quadratic terms we deduce that either all roots are real for all positive values of the remaining parameters if the quadratic terms come from black edges, or in case of red edges we have the simple explicit inequality $(\xi_i + \xi_j)^2 > 16\xi_i\xi_j$. Moreover if all the roots are different as it happens in most cases we have that in an open neighborhood of this set the roots are still real and distinct. So we have to make sure that all these neighborhoods have a non-empty intersection.

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